PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

A Thermodynamical Theory of Turbulence I. Basic Developments

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Phil. Trans. R. Soc. Lond. A 1989 **327**, 415-448
doi: 10.1098/rsta.1989.0001

Phil. Trans. R. Soc. Lond. A **327**, 415–448 (1989) [415]
Printed in Great Britain

A THERMODYNAMICAL THEORY OF TURBULENCE
I. BASIC DEVELOPMENTS

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(*Communicated by A. E. Green, F.R.S. – Received* 24 *August* 1987 – *Revised* 4 *December* 1987)

CONTENTS

This paper is concerned with the construction of a thermodynamical theory for turbulence based on a continuum model consistent with a wide range of experimental results and observations. A complete theory with appropriate constitutive equations is developed for viscous turbulent flow but the special case of (rate-independent) inviscid turbulent flow is also discussed. The theoretical results obtained readily account for such mechanical aspects of turbulent flow as anisotropy, as well as the energetic effects of turbulent fluctuations, in addition to the more standard thermomechanical effects. More specifically, three different scales of motion and modelling, namely molecular, microscopic and macroscopic, are considered in the construction of the basic theory. Whereas the ordinary *thermal effects* (such as temperature) on the macroscopic scale represent the manifestation of vibratory motions at the molecular level, similar variables are used to represent the energetic *turbulent effects* on the macroscopic level that arise from turbulent fluctuations at the microscopic level. The various ingredients of the thermodynamical aspects (both due to thermal and turbulent effects) of the continuum model are incorporated into the theory by means of a recent procedure to thermodynamics by Green & Naghdi (*Proc.*

R. Soc. Lond. A **357**, 253 (1977)). The mechanical aspects of the model for a turbulent fluid requires admission of additional balance laws for eddy concentration and for a kinematical variable which represents the effect of alignment of these eddies (at the microscopic level) along a particular direction on the macroscopic scale, in accordance with observations by Townsend (*The structure of turbulent shear flow*, Cambridge University Press (1976)) and others.

1. Introduction

Numerous experimental results reported in the turbulence literature of the past four decades appear to indicate that turbulent flow is strongly influenced by the existence, and alignment in certain preferred directions, of a certain class of 'large' eddies or vortices that contain most of the energy associated with turbulent fluctuations. This class of eddies is referred to as 'large eddies' in some of the older literature, but in the more recent literature (for example, Townsend (1976) and Savill (1987)) is identified as the 'main turbulent motion' or 'roller eddies'. Recent interest in the structure of large eddies of this class and in their interaction with the so-called 'mean' flow has produced a substantial volume of literature. The purpose of the present paper is to construct a manageable theory of turbulence which accommodates both the effect of alignment of these large eddies and the energetic effects of turbulent fluctuations, as well as the usual thermomechanical effects.

Many theories of turbulence employ a constitutive relationship for the 'mean' stress similar to that of a linear viscous fluid in laminar motion and allow the viscosity coefficients to depend upon a set of additional parameters (such as 'turbulent energy', 'mixing length', 'pseudo-vorticity', 'turbulent dissipation', etc.), and all of these are intended to describe some details of the turbulent motion. In the early studies of Taylor (1915) and Prandtl (1925) and more recently in the work of Cebeci & Smith (1974), these additional variables are simply prescribed for a particular flow. Several later studies (for example, Saffman 1970; Bradshaw 1972; Lumley 1970; Launder *et al.* 1975) assign a certain 'model equation' to each of these additional variables in the hope of obtaining equations valid for all flow-field geometries. Although models employing this type of stress constitutive equations have produced some interesting results in certain special cases, they have not been able to properly account for the 'anisotropic' structure of turbulence. Several fairly recent developments (such as 'stress-equation' modelling, modern 'rapid-distortion' theory, and 'large-eddy simulations') attempt to correct for this shortcoming in various ways; however, these studies tend either to be extremely *ad hoc* or to depend heavily on a specified computational procedure (see reviews by Reynolds (1976) and Savill (1987)). Although computer simulations of the large eddy movements using simplified models of the small-scale eddies are yielding increasingly interesting and useful results, the need for a complete physical theory of turbulence that incorporates the effect of alignment of the large eddies, as well as the energetic effects of turbulent fluctuations, still exists. Such a theory, as discussed here, must necessarily include a thermomechanical procedure by which appropriate conservation laws and constitutive equations for the additional turbulence parameters may be introduced in full agreement with the requirements of continuum physics and a given concept of the microscopic flow structure.

1.1. *Scales of motion and modelling: macroscopic, microscopic and molecular scales*

Before a discussion of the contents of the paper, we first describe various scales of motion that will be referred to throughout the paper. It is common in the study of ordinary fluids to model the behaviour of the fluid as a macroscopic continuum, although on a much finer scale the same fluid may be viewed as an aggregate of discrete molecules. It is conceivable that the same equations which represent the macroscopic motion in one régime may describe motion on a scale which in some sense is 'microscopic' in another régime, the designation 'macroscopic' being here reserved for an even coarser scale of modelling. This type of situation indeed occurs in regard to the usual Navier–Stokes equations when crossing over from the laminar to the turbulent régimes. Thus, for a turbulent flow, we admit three distinct scales of motion – macroscopic, microscopic and molecular – such that in the laminar flow régime the macroscopic and microscopic scales are equivalent. To further illustrate these levels of physical modelling, consider the flow of a typical fluid, water say, in a straight circular pipe. Near the mouth of the pipe, the fluid is laminar; and, by modelling the fluid as a continuum, solutions for the primary quantities of interest (such as velocity and shear stresses) can be readily obtained. Here any model of the fluid on a molecular scale is intended to serve as a 'background' model for the continuum theory. By this we mean that although certain features of the molecular motion may be represented in the continuum theory, such as the representation of the intensity of molecular vibrations by temperature, one is not interested in every detail of the molecular motion, i.e. one is not interested in the location of every molecule for all times. Alternatively, one may attempt to solve for the aforementioned quantities of interest directly from a molecular model of the fluid. This latter approach is, of course, in general neither expedient nor practical if indeed it is even possible. As we progress further along the pipe, the fluid becomes unsteady to small changes in velocity and eventually becomes turbulent. In a turbulent fluid, it is no longer straightforward to obtain solutions for the primary quantities of interest from a continuum theory that is represented only by the Navier–Stokes equations. This is because the flow is immensely complicated by the presence of a large number of eddies and vortices. Furthermore, the applications to which we usually apply turbulent modelling no longer require that we know the velocity at every point in the fluid or the shear stresses on the pipe wall for every instant of time, but rather some mean value of these variables. (This is not to say, of course, that knowledge of the instantaneous values of these quantities would not enhance our understanding of the basic mechanisms of the turbulent fluctuations.) At this point in our discussion, it is advantageous to make the scale of our physical model somewhat coarser so that the relevant equations resulting from the model can be solved directly for the quantities of interest. Thus, we define the *macroscopic level* as the scale of motion which describes in some sense the 'bulk' or 'mean' features of the flow. (The word 'mean' here is not intended in a strictly mathematical sense. In view of the additional kinematical and thermal structure to be introduced presently, the macroscopic field variables as defined here are not necessarily the same as the usual 'mean' flow values of the classical variables.) At times, however, it may become necessary to refer back to our original scale containing turbulent eddies and vortices. We, therefore, define the *microscopic level* as the scale of motion in which turbulent fluctuations of the continuum are occurring. Although the microscopic level may be well described by the standard Navier–Stokes equations, new conservation and constitutive equations must be found for the macroscopic level. In laminar flow, of course, the microscopic and macroscopic levels become coincident, so the governing equations of the desired macroscopic turbulent theory must contain the Navier–Stokes equations as a special case.

Before leaving this discussion on basic approaches to turbulence modelling, it may be of value to briefly contrast the approach of the present theory with those usually used in studies of turbulence. Perhaps the most common type of turbulence modelling involves statistical averaging of the microscopic flow equations (i.e. the usual 'Reynolds averaging'). Despite their long usage in the literature, statistical studies have produced relatively few results of general interest and suffer from the well-known 'closure' problem. Also, it is very difficult in statistical studies to identify the effects of the underlying microscopic structures which influence the macroscopic turbulent flow. In response to this latter difficulty, many theoretical studies have recently turned to the construction and solution of certain idealized (laminar) flows that seem similar to such microscopic turbulence structures. These studies then attempt to somehow combine the idealized flow structures to form governing equations for the macroscopic flow; however, the manner in which the microscopic solutions are combined varies greatly, and the resulting macroscopic equations often bear a closer resemblance to a computational scheme than to the governing equations of a physical theory. The present paper represents an attempt to formulate a manageable physical theory of macroscopic turbulent flow. This theory has the ability to incorporate many of the previously mentioned idealized microscopic solutions; and, indeed, such a procedure is desirable in order to motivate physically realistic forms of constitutive equations for certain of the dependent variables.

1.2. *Additional independent variables for turbulence*

As will become evident later, the turbulent fluid in the present paper is modelled directly on the macroscopic level and both the microscopic and molecular levels are treated as background models. However, certain independent variables are admitted in the macroscopic model to describe particular features of phenomena occurring at the microscopic and molecular levels. A classic example of such a variable is temperature, which is commonly used in continuum theories to describe the intensity of the chaotic, or 'hidden,' motion of the molecules. (The terminology of 'hidden' motion here refers to motion in one or more background models which is not represented by the macroscopic kinetic energy.) In turbulent flow, such 'hidden' motions occur both on the molecular and microscopic levels, so two temperature-like variables are needed. These temperatures, designated as the 'thermal temperature' and the 'turbulent temperature', describe the respective intensities of molecular vibrations and turbulent fluctuations. Additionally, certain structures arising from turbulent fluctuations at the microscopic level are thought to influence the macroscopic responses, giving rise to the so-called 'anisotropic' effects often observed in turbulent flow experiments. The existence of these structures and the means by which they affect the 'mean' flow in the turbulent régime was demonstrated by Townsend (1956, 1976) and Grant (1958), among others. A list of quotations from various sources which lend support for the model presented in §§4 and 5 is collected in Appendix A.

Certain microscopic structures possessing directional dependence are represented on a macroscopic scale by the introduction of a single additional vector-valued quantity, called *director*. Background information concerning 'directed' or 'oriented' media (also called Cosserat continua), which utilize one or more directors may be found in several papers included in the list of references, for example, Ericksen (1961), Green *et al.* (1965), Leslie (1968), Green & Naghdi (1976*a*, *b*), Naghdi (1982), as well as in Truesdell & Noll (1965), Naghdi (1972) and Chandrasekhar (1977). As noted in the preceding paragraph, two temperature-like variables are used in our characterization of turbulent model. In this connection, it should be noted that the notion of two or more temperatures has been utilized previously in different contexts for thermoelastic shells (Green & Naghdi 1979*b*) and for mixtures with different constituent temperatures (Green & Naghdi 1978).

Returning to the description of the model used, in conjunction with the director and the notion of two temperatures, we also admit N additional scalar-valued quantities and associated balance equations in the development of the general theory. Two of these quantities are identified as 'thermal entropy' and as 'turbulent entropy'. Using a recent approach to thermodynamics developed by Green & Naghdi (1977*a*), we find that the existence of two distinct entropies is demanded by the assumption of two temperatures, such that the entropy balance laws provide sufficient equations for the determination of the temperature fields. Another of these scalar-valued quantities is identified as the 'eddy density', i.e. the number of eddies of a certain class per unit volume of the fluid. The eddy density (as defined in §5) can be related to the 'mixing length' of Prandtl (1925) or, in conjunction with the turbulent temperature, to the 'turbulence dissipation' parameter used by Launder *et al.* (1975) or the 'psuedo-vorticity' scalar used by Saffman (1970).

1.3. *A description of the contents of the paper*

In §2 of the paper a fairly rapid summary is given for a continuum theory of a directed medium endowed with a single director and N additional scalar quantities. Here the balance laws are first stated in integral form, as they are needed in the subsequent derivation of the jump conditions over a surface of discontinuity in §3. These jump conditions are more general than those usually presented in the three-dimensional continuum mechanics literature in that supply of various fundamental quantities are also admitted at the surface of discontinuity. The thermodynamics pertinent to a discussion of turbulent fluid flow are developed in §4. It is here that the thermal and turbulent temperatures, as well as their associated entropies, are explicitly introduced. A specific model for the mechanical aspect of a turbulent fluid, based upon the observations of Townsend (1956, 1976) and others (see Appendix A), is presented in §5 and incorporates an additional balance law for eddy density. Further, because this model requires that a certain constraint be imposed on the director, the forms of the constraint response functions are developed in the rest of this section. The treatment of the constrained director follows the approach of Green *et al.* (1970; §6).

The restrictions on constitutive equations for viscous turbulent flow (for both compressible and incompressible fluids), apart from those arising from considerations of invariance under superposed rigid body motions, are obtained in accordance with the recent approach to thermodynamics of deformable media by Green & Naghdi (1977*a*) in §6. Although additional restrictions may be imposed from the Second Law of Thermodynamics, they are not discussed in the present paper. In §7, the jump conditions developed in §3 are applied to the special case of the interface separating a turbulent fluid and a non-turbulent material. In §8, the more general theory of §§5 and 6 is simplified for a (rate-independent) inviscid fluid, both for compressible and incompressible cases. Finally, some additional remarks are provided in §9 concerning the main features of the theory proposed in this paper, along with some comments that are intended to clarify the importance of invariance requirements under superposed rigid body motions. The latter comments are included here only because the validity of invariance requirements has been questioned in some of the recent literature on turbulence.

2. Basic concepts; conservation laws

As noted in §1, the purely mechanical aspects of our development are formulated in the context of the three-dimensional theory of directed media, and we only need to utilize a single director. The thermodynamical aspects of the subject as presented here follow the procedure of Green & Naghdi (1977*a*) who, in addition to the balance of energy, also introduced a balance of entropy. This balance of entropy leads to a scalar field equation, which, in particular, involves the rate of entropy and the rate of internal generation of entropy, as well as the rate of heat supply and the flux of entropy (see Green & Naghdi 1977*a*, equations (2.3) and (2.6)). As will become evident in §§4 and 5, for our continuum model of turbulence we require still two other balance laws involving scalar fields that are symbolically analogous to the entropy balance of Green & Naghdi. Because of this, as well as other considerations mentioned in §§4 and 5, it is convenient to record a general balance law for N scalar fields instead of one or more separate scalar balance laws.

We consider here a finite body $\mathscr{B}$ – in the context of directed media – with each of its material points (or particles) X being endowed with an additional independent kinematical vector field, namely a director. Thus, let the material point X and the director at X, be identified by the place $\boldsymbol{X}$ and the value of the single director $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{X})$ in a fixed reference configuration κ_R; and, similarly, denote the corresponding quantities in the current configuration κ of the body at time t by the place $\boldsymbol{x}$ and the director $\boldsymbol{d} = \boldsymbol{d}(\boldsymbol{D}, t)$ at $\boldsymbol{x}$. A motion of such a body is then defined by sufficiently smooth vector functions $\boldsymbol{\chi}$ and $\mathscr{D}$ which assign the place $\boldsymbol{x}$ and director $\boldsymbol{d}$ to each material point of $\mathscr{B}$ at each instant of time, i.e.

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t), \quad \boldsymbol{d} = \mathscr{D}(\boldsymbol{D}, t). \tag{2.1}$$

Clearly, in view of the dependence of the reference director $\boldsymbol{D}$ on $\boldsymbol{X}$, the right-hand side of $(2.1)_2$ can be expressed as a different function of $\boldsymbol{X}, t$. We assume that $(2.1)_1$, but not $(2.1)_2$, is invertible for a fixed value of t so that the jacobian of transformation associated with $(2.1)_1$ does not vanish; for definiteness, however, we stipulate the jacobian $\det \boldsymbol{F} > 0$, where $\boldsymbol{F} = \partial\boldsymbol{\chi}/\partial\boldsymbol{X}$. Also, the ordinary particle velocity $\boldsymbol{v}$ and the director velocity $\boldsymbol{w}$ are defined by

$$\boldsymbol{v} = \dot{\boldsymbol{x}}, \quad \boldsymbol{w} = \dot{\boldsymbol{d}}, \tag{2.2}$$

where a superposed dot denotes material time differentiation with respect to t holding $\boldsymbol{X}$ fixed.

In the present configuration, the body $\mathscr{B}$ bounded by a closed surface $\partial\mathscr{B}$ occupies a region of space $\mathscr{R}$ bounded by a closed surface $\partial\mathscr{R}$. Similarly, in the present configuration, an arbitrary material volume of $\mathscr{B}$ occupies a portion of $\mathscr{R}$ designated here by $\mathscr{P}(\subseteq \mathscr{R})$ and bounded by a closed surface $\partial\mathscr{P}$. The outward unit normal to this surface is $\boldsymbol{n}$.

It is convenient at this point to define certain additional quantities which occur in the balance laws: the mass density $\rho = \rho(\boldsymbol{X}, t)$ of $\mathscr{B}$ in the current configuration; the stress vector $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{X}, t; \boldsymbol{n})$ and the director stress (or the stress couple) vector $\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{X}, t; \boldsymbol{n})$, each measured per unit area in the current configuration; the external body force $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{X}, t)$ and the external director body force $\boldsymbol{l} = \boldsymbol{l}(\boldsymbol{X}, t)$, each per unit mass; the intrinsic director force $\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{X}, t)$ per unit volume in the current configuration; the specific internal energy $\epsilon = \epsilon(\boldsymbol{X}, t)$; the heat flux $h = h(\boldsymbol{X}, t; \boldsymbol{n})$ per unit time and measured per unit area in the current configuration; and the heat supply $r = r(\boldsymbol{X}, t)$ per unit time and per unit mass. In addition, we need to introduce scalar-valued fields $\eta^N = \eta^N(\boldsymbol{X}, t)$ per unit mass $(N = 1, 2, \ldots, K)$, an internal rate of supply (or production) $\xi^N = \xi^N(\boldsymbol{X}, t)$ per unit mass, the external rate of supply $s^N = s^N(\boldsymbol{X}, t)$ and the internal surface flux $k^N = k^N(\boldsymbol{X}, t; \boldsymbol{n})$ measured per unit area of the surface $\partial\mathscr{P}$. It should be emphasized that the dependence of the scalar fields k^N on the outward unit normal $\boldsymbol{n}$ is similar to the dependence of the heat flux h and the vectors $\boldsymbol{t}$ and $\boldsymbol{m}$ on $\boldsymbol{n}$.

We also assume the kinetic energy per unit mass for the directed medium under consideration to have the form

$$\kappa = \tfrac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{v} + 2y_1\boldsymbol{v}\cdot\boldsymbol{w} + y_2\boldsymbol{w}\cdot\boldsymbol{w}), \tag{2.3}$$

where the inertia coefficients $y_1 = y_1(\boldsymbol{X})$ and $y_2 = y_2(\boldsymbol{X})$ are independent of time and in general require constitutive equations. In view of (2.3), we define the momentum per unit mass corresponding to the velocity $\boldsymbol{v}$ and the director momentum per unit mass corresponding to the director velocity $\boldsymbol{w}$ by

$$\partial\kappa/\partial\boldsymbol{v} = \boldsymbol{v} + y_1\boldsymbol{w}, \quad \partial\kappa/\partial\boldsymbol{w} = y_1\boldsymbol{v} + y_2\boldsymbol{w}. \tag{2.4}$$

Also the physical dimensions of ρ, $\boldsymbol{t}$, $\boldsymbol{b}$ are

$$\left.\begin{array}{l}\text{phys. dim. } \rho = [\mathrm{ML^{-3}}], \quad \text{phys. dim. } \boldsymbol{t} = [\mathrm{ML^{-1}T^{-2}}],\\ \text{phys. dim. } \boldsymbol{b} = [\mathrm{LT^{-2}}],\end{array}\right\} \tag{2.5}$$

where the symbols [L], [M] and [T] stand for the physical dimensions of length, mass and time. The dimensions of the vector fields $\boldsymbol{m}$, $\boldsymbol{l}$, $\boldsymbol{k}$ depend upon the physical dimension that one may assume for $\boldsymbol{d}$. Here we assume that $\boldsymbol{d}$ has the dimension of length and then $\boldsymbol{m}$, $\boldsymbol{l}$ will have the same physical dimensions as $\boldsymbol{t}$, $\boldsymbol{b}$ in (2.5) while $\boldsymbol{k}$ will have the physical dimension of $[\mathrm{ML^{-2}T^{-2}}]$. {It should be noted that if $\boldsymbol{d}$ is specified to be dimensionless, then the physical dimension of $\boldsymbol{m}$ will be $[\mathrm{MT^{-2}}]$ corresponding to the physical dimension of a stress couple.}

In terms of the above definitions of the various field quantities and with reference to the present configuration, the various conservation laws for any part $\mathscr{P}$ of a directed body under discussion are:

$$\left.\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\rho\,\mathrm{d}v = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\rho(\boldsymbol{v}+y_1\boldsymbol{w})\,\mathrm{d}v = \int_{\mathscr{P}}\rho\boldsymbol{b}\,\mathrm{d}v + \int_{\partial\mathscr{P}}\boldsymbol{t}\,\mathrm{d}a,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\rho(y_1\boldsymbol{v}+y_2\boldsymbol{w})\,\mathrm{d}v = \int_{\mathscr{P}}(\rho\boldsymbol{l}-\boldsymbol{k})\,\mathrm{d}v + \int_{\partial\mathscr{P}}\boldsymbol{m}\,\mathrm{d}a,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\rho[\boldsymbol{x}\times(\boldsymbol{v}+y_1\boldsymbol{w})+\boldsymbol{d}\times(y_1\boldsymbol{v}+y_2\boldsymbol{w})]\,\mathrm{d}v = \int_{\mathscr{P}}\rho(\boldsymbol{x}\times\boldsymbol{b}+\boldsymbol{d}\times\boldsymbol{l})\,\mathrm{d}v + \int_{\partial\mathscr{P}}(\boldsymbol{x}\times\boldsymbol{t}+\boldsymbol{d}\times\boldsymbol{m})\,\mathrm{d}a,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\rho(\epsilon+\kappa)\,\mathrm{d}v = \int_{\mathscr{P}}\rho(r+\boldsymbol{b}\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{w})\,\mathrm{d}v + \int_{\partial\mathscr{P}}(\boldsymbol{t}\cdot\boldsymbol{v}+\boldsymbol{m}\cdot\boldsymbol{w}-h)\,\mathrm{d}a,
\end{aligned}\right\} \tag{2.6}$$

where $\mathrm{d}v$ is an element of volume and $\mathrm{d}a$ is an element of area in the current configuration. The first of (2.6) is a mathematical statement of conservation of mass, the second that of the linear momentum, the third that of the director momentum, the fourth represents the conservation of moment of momentum and the fifth is the law of conservation of energy.

As will become evident in §§4 and 5, we need to introduce additional balance laws which involve only scalar fields of the type η^N, ξ^N, s^N, k^N introduced in the preceding paragraph. We postpone (until §§4 and 5) the exact specification of the physical significance of these scalar fields and for the moment regard these variables to be scalars having the same invariance properties as the scalars r, h in $(2.6)_5$ and other scalar variables occurring in the entropy balance law of Green & Naghdi (1977*a*, equation (2.3)), i.e. these scalars remain unaltered if the medium is at any time subjected to a superposed rigid body translation and rigid body rotation. With this background, for later use we now state a general balance law of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho\eta^N \,\mathrm{d}v = \int_{\mathscr{P}} \rho(s^N+\xi^N)\,\mathrm{d}v - \int_{\partial\mathscr{P}} k^N \,\mathrm{d}a \quad (N=1,2,\dots,K). \tag{2.7}$$

It is clear that (2.7) represents N scalar balance laws. For $K=1$, it will be identified later in §4 with the balance of entropy utilized previously by Green & Naghdi (1977*a*).

By usual procedures and under suitable continuity assumptions, it follows from $(2.6)_{2,3,5}$ and (2.7) that

$$\boldsymbol{t} = \boldsymbol{t}_i n_i, \quad \boldsymbol{m} = \boldsymbol{m}_i n_i, \quad h = q_i n_i, \quad k^N = p_i^N n_i, \tag{2.8}$$

where n_i are the components of the outward unit normal $\boldsymbol{n}$ referred to a set of right-handed orthonormal base vectors $\boldsymbol{e}_i$ associated with a rectangular cartesian coordinate system and $\boldsymbol{t}_i$, $\boldsymbol{m}_i$, q_i, p_i^N are only functions of $\boldsymbol{X}, t$ independent of $\boldsymbol{n}$. Referred to $\boldsymbol{e}_i$, the various coefficients of n_i in (2.8) may be expressed as

$$\boldsymbol{t}_i = t_{ji}\boldsymbol{e}_j, \quad \boldsymbol{m}_i = m_{ji}\boldsymbol{e}_j, \quad \boldsymbol{q} = q_i\boldsymbol{e}_i, \quad \boldsymbol{p}^N = p_i^N\boldsymbol{e}_i, \tag{2.9}$$

where t_{ji} are the components of the stress tensor, m_{ji} are the components of the director stress (or the couple stress) tensor, q_i are the components of the heat flux vector $\boldsymbol{q}, p_i^N$ are the components of the flux vector $\boldsymbol{p}^N$ associated with k^N, all Latin indices take the values 1, 2, 3 and summation over repeated Latin indices is understood.

With the use of the results (2.8) and (2.9), from the six equations in (2.6) and (2.7) follow the local forms of the conservation laws, which can be displayed as

$$\left.\begin{aligned} &\dot{\rho}+\rho\,\mathrm{div}\,\boldsymbol{v}=0,\\ &\rho(\dot{\boldsymbol{v}}+y_1\dot{\boldsymbol{w}})=\rho\boldsymbol{b}+\boldsymbol{t}_{i,i},\\ &\rho(y_1\dot{\boldsymbol{v}}+y_2\dot{\boldsymbol{w}})=\rho\boldsymbol{l}-\boldsymbol{k}+\boldsymbol{m}_{i,i},\\ &\boldsymbol{e}_i\times\boldsymbol{t}_i+\boldsymbol{d}_{,i}\times\boldsymbol{m}_i+\boldsymbol{d}\times\boldsymbol{k}=0,\\ &\rho\dot{\varepsilon}=\rho r-\mathrm{div}\,\boldsymbol{q}+P, \end{aligned}\right\} \tag{2.10}$$

and

$$\rho\dot{\eta}^N = \rho(s^N+\xi^N)-\mathrm{div}\,\boldsymbol{p}^N, \quad (N=1,2,\dots,K), \tag{2.11}$$

where comma denotes partial differentiation, div stands for the divergence operator with respect to the place $\boldsymbol{x}$ keeping t fixed and the mechanical power P in $(2.10)_5$ is defined by

$$P = \boldsymbol{t}_i\cdot\boldsymbol{v}_{,i}+\boldsymbol{m}_i\cdot\boldsymbol{w}_{,i}+\boldsymbol{k}\cdot\boldsymbol{w} = t_{ji}v_{j,i}+m_{ji}w_{j,i}+k_i w_i. \tag{2.12}$$

It is desirable to indicate a procedure for the utilization of the balance laws (2.6) and (2.11) for $N=1,2,3,$ and the relevant constitutive equations for turbulent viscous fluids. We postpone a description of such a procedure until the end of §5, but include here a few remarks pertaining to invariance under superposed rigid body motions. We recall that as a consequence of the motion specified by the vector functions χ and $\mathscr{D}$ the body occupies a configuration $\boldsymbol{\kappa}$ at time t; and, in this configuration, the place $\boldsymbol{x}$ and the director $\boldsymbol{d}$ at $\boldsymbol{x}$ are given by $(2.1)_{1,2}$.

Under another motion, which differs from the given one only by a superposed rigid body motion, the material point X and the director at X move, respectively, to $\boldsymbol{x}^+$ and $\boldsymbol{d}^+$ in the configuration $\boldsymbol{\kappa}^+$ at time $t^+ = t+a$, where a is a constant. It is well known that under such superposed rigid body motions the place $\boldsymbol{x}^+$ and the director $\boldsymbol{d}^+$ at $\boldsymbol{x}^+$ are specified by

$$\boldsymbol{x}^+ = \boldsymbol{a}+\boldsymbol{Q}\boldsymbol{x}, \quad \boldsymbol{d}^+ = \boldsymbol{Q}\boldsymbol{d}, \tag{2.13}$$

where $\boldsymbol{a}$ is a vector function of t and $\boldsymbol{Q}$ is a proper orthogonal tensor function of t. The vector $\boldsymbol{a}$ in $(2.13)_1$ can be interpreted as a rigid body translation and $\boldsymbol{Q}$ as a rigid body rotation. Also the specification $(2.13)_2$ implies that the magnitude of $\boldsymbol{d}$ remains unchanged under superposed rigid body motion $(2.13)_1$. For additional related background we refer to Green & Naghdi (1979*a*) and to Naghdi (1972). Now, all conservation equations in (2.10) and (2.11) and the various fields occurring in these equations are regarded to be properly invariant under superposed rigid body motions (2.13): for example, the scalar r in $(2.10)_5$ and the fields k_i and t_{ij} in (2.12) transform according to the formulae

$$r^+ = r, \quad k_i^+ = Q_{im}\,k_m, \quad t_{ij}^+ = Q_{im}\,Q_{jn}\,t_{mn}, \tag{2.14}$$

where Q_{ij} are the cartesian components of the proper orthogonal tensor $\boldsymbol{Q}$. We may refer to scalars, vectors or second-order tensors that obey transformations of the form (2.14) as *objective*; but we emphasize that our use of the term objective differs from the corresponding usage by some authors who appeal to the 'principle of material frame-indifference' and allow $\boldsymbol{Q}$ to be any orthogonal tensor. (For additional related background and discussion see Green & Naghdi (1979*a*).)

3. Discontinuity conditions

In the previous section no surface of discontinuity was admitted and the local forms (2.10) and (2.11) of the conservation laws were deduced under usual continuity assumptions. In this section we provide a brief derivation of the jump conditions across a surface of discontinuity, associated with each of the six equations in (2.6) and (2.7), which include an extension of the standard jump conditions in the three-dimensional theory. A similar extension, but using a somewhat different approach, was carried out previously in the context of the theory of a directed fluid sheet and was discussed with various degrees of generality in the papers of Caulk (1976), Green & Naghdi (1977*b*) and Naghdi & Rubin (1981). The extension just referred to involves admitting a primitive notion that the surface of discontinuity itself can also act as a source, thereby providing a supply term on the right-hand side of each of the balance laws in (2.6) and (2.7).

Consider a surface of discontinuity $\Sigma(t)$ in the present configuration of $\mathscr{B}$ and let $\sigma(\subseteq \Sigma)$, with a closed boundary curve $\partial\sigma$, be an arbitrary portion of the surface of discontinuity contained in the material part $\mathscr{P}(\subseteq \mathscr{R})$. Let $\boldsymbol{\nu}$ be the unit normal to Σ chosen in some specified direction and denote by U the normal velocity of Σ in the direction of $\boldsymbol{\nu}$. Then, the relative velocity $\boldsymbol{u}$ of Σ is defined by

$$\boldsymbol{u} = \boldsymbol{v} - U\boldsymbol{\nu}. \tag{3.1}$$

It is convenient to recall here the transport theorem which is needed in the derivation of the desired jump conditions from the balance laws in (2.6) and (2.7). Thus, let ϕ be any function of position and time that takes on different values ϕ_2 and ϕ_1 on either side of Σ. Then, a statement of the transport theorem is (see Truesdell & Toupin 1960, equation (192.4)):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \phi \,\mathrm{d}v = \int_{\mathscr{P}} (\dot{\phi} + \phi \operatorname{div} \boldsymbol{v})\,\mathrm{d}v + \int_{\sigma} [\![\phi \boldsymbol{u}\cdot\boldsymbol{\nu}]\!]\,\mathrm{d}a \tag{3.2}$$

where we have used the notation $[\![\phi]\!] = \phi_2 - \phi_1$.

When the region $\mathscr{P}$ contains a part σ of the surface of discontinuity, for our derivation of the extended version of the jump conditions, we need to admit a *surface supply* term to be added to the right-hand side of each of the integral balance equations in (2.6) and (2.7). These surface supply terms in the order of appearance of the integral balance laws $(2.6)_{1-5}$ and (2.7) are:

$$\int_{\sigma} M\,\mathrm{d}a,\quad \int_{\sigma} \boldsymbol{F}\,\mathrm{d}a,\quad \int_{\sigma} \boldsymbol{L}\,\mathrm{d}a,\quad \int_{\sigma} \boldsymbol{M}\,\mathrm{d}a,\quad \int_{\sigma} \Phi\,\mathrm{d}a,\quad \int_{\sigma} E^N\,\mathrm{d}a, \tag{3.3}$$

whose integrands in general require constitutive equations. In (3.3), the scalar M stands for the surface mass supply, $\boldsymbol{F}$ and $\boldsymbol{L}$ are the ordinary and director surface momentum supplies, $\boldsymbol{M}$ is the surface supply of moment of momentum, Φ is the surface supply of energy and E^N, $(N = 1, 2, \ldots, K)$ are the surface supplies of the scalar quantities associated with (2.7), all per unit area of the surface Σ.

In order to illustrate the nature of the extended forms of the jump conditions, we include here a brief derivation of the first two and then merely record the remaining jump conditions. Consider first the extended version of the conservation of mass in the presence of the surface supply term $(3.3)_1$. With the use of the transport theorem (3.2) and the definition (3.1), this can be written as

$$\int_{\mathscr{P}} (\dot{\rho} + \rho \operatorname{div} \boldsymbol{v})\,\mathrm{d}v + \int_{\sigma} [\![\rho \boldsymbol{u}\cdot\boldsymbol{\nu}]\!]\,\mathrm{d}a = \int_{\sigma} M\,\mathrm{d}a.$$

In view of $(2.10)_1$, the first term in the above expression vanishes and we are left with surface integrals which hold for an arbitrary part σ. Hence, we may deduce that

$$[\![\rho \boldsymbol{u}\cdot\boldsymbol{\nu}]\!] = M, \tag{3.4}$$

as the jump condition for mass at the surface of discontinuity. Again, with the use of the transport theorem (3.2) and the divergence theorem, the balance of linear momentum in the presence of surface supply term $(3.3)_2$ may be reduced to

$$\int_{\mathscr{P}} \{\rho(\dot{\boldsymbol{v}} + y_1 \dot{\boldsymbol{w}}) + (\dot{\rho} + \rho \operatorname{div} \boldsymbol{v})\,(\boldsymbol{v} + y_1 \boldsymbol{w})\}\,\mathrm{d}v + \int_{\sigma} [\![\rho \boldsymbol{u}\cdot\boldsymbol{\nu}(\boldsymbol{v} + y_1 \boldsymbol{w})]\!]\,\mathrm{d}a$$
$$= \int_{\mathscr{P}} \{\rho \boldsymbol{b} + \boldsymbol{t}_{i,i}\}\,\mathrm{d}v + \int_{\sigma} [\![\boldsymbol{t}]\!]\,\mathrm{d}a + \int_{\sigma} \boldsymbol{F}\,\mathrm{d}a.$$

In view of $(2.10)_2$, the last equation can be reduced to integrals which hold for an arbitrary σ and we may deduce the result

$$[\![\rho \boldsymbol{u}\cdot\boldsymbol{\nu}(\boldsymbol{v} + y_1 \boldsymbol{w}) - \boldsymbol{t}]\!] = \boldsymbol{F}, \tag{3.5}$$

as the jump condition for momentum at the surface of discontinuity. The remaining jump conditions can be derived similarly and are given by

$$[\![\rho \boldsymbol{u}\cdot\boldsymbol{\nu}(y_1 \boldsymbol{v} + y_2 \boldsymbol{w}) - \boldsymbol{m}]\!] = \boldsymbol{L}, \tag{3.6}$$

$$[\![\boldsymbol{d} \times \{\rho \boldsymbol{u}\cdot\boldsymbol{\nu}(y_1 \boldsymbol{v} + y_2 \boldsymbol{w}) - \boldsymbol{m}\}]\!] = \boldsymbol{M} - \boldsymbol{x} \times \boldsymbol{F}, \tag{3.7}$$

$$[\rho(\epsilon+\kappa)\,\boldsymbol{u}\cdot\boldsymbol{v}-\boldsymbol{t}\cdot\boldsymbol{v}-\boldsymbol{m}\cdot\boldsymbol{w}+\boldsymbol{q}\cdot\boldsymbol{v}] = \Phi, \tag{3.8}$$

$$[\rho\eta^N(\boldsymbol{u}\cdot\boldsymbol{v})+\boldsymbol{p}^N\cdot\boldsymbol{v}] = E^N, \quad (N = 1,2,\ldots,K). \tag{3.9}$$

As noted earlier, the surface supply (or source) terms on the right-hand sides of the jump conditions (3.4)–(3.9) in general must be specified by constitutive equations. Also, it should be noted that for a directed medium with surface supplies at the surface of discontinuity, the jump in moment of momentum (3.7) is not necessarily identically satisfied as is its counterpart in the standard (classical) statements of jumps. In the special case that $\boldsymbol{d}$ is continuous across the surface of discontinuity, with the help of (3.6), the jump in moment of momentum (3.7) reduces to

$$\boldsymbol{M} = \boldsymbol{x}\times\boldsymbol{F}+\boldsymbol{d}\times\boldsymbol{L}. \tag{3.10}$$

It is of interest to note that in the absence of surface supply terms $\boldsymbol{F}$, $\boldsymbol{L}$, and $\boldsymbol{M}$, the jump in moment of momentum requires that

$$[\boldsymbol{d}] = 0, \tag{3.11}$$

and hence $\boldsymbol{d}$ will be continuous in that case. Of course, if any one or more of the surface supply terms $\boldsymbol{F}$, $\boldsymbol{L}$, $\boldsymbol{M}$ are non-zero, $\boldsymbol{d}$ may be either continuous or discontinuous across the surface Σ.

In many physical problems, especially those involving interfaces between materials of different microscopic structure, it is desirable to allow the director to be discontinuous across a surface of discontinuity. Such a situation, which may give rise to a dilemma, justifies the inclusion of surface supply terms in the general statements of the jump conditions. Moreover, the various supply terms must be such that each of the jump conditions (3.4)–(3.9) are properly invariant under superposed rigid body motions. This, in turn, places restrictions on the manner of dependence of the constitutive response functions for the surface supply terms on the independent variables of the particular flow field.

4. Thermodynamical background for the continuum model of turbulence proposed in §5

In classical thermodynamics, temperature is regarded as a measure of the average deviation of the molecular motion from the mean motion. Although molecular vibration is 'hidden' on a macroscopic scale, its manifestation on the macroscopic scale alters the internal energy and other dependent thermal variables. Such 'hidden' motions in a turbulent fluid occur both on the small molecular scale and on the much larger scale of the turbulent eddies. Thus, corresponding to these notions, we admit here two temperatures: a 'thermal temperature' θ_{H} and a 'turbulent temperature' θ_{T}, which, respectively, serve as measures of the hidden motion on the molecular and the microscopic scales. Both of these temperatures are assumed to be absolute; and, in particular, the vanishing of the turbulent temperature θ_{T} implies the cessation of *all* turbulent motions.

Throughout this section and in the remainder of the paper, we attach a subscript H or T to variables identified as 'thermal' or 'turbulent', respectively. Also, the tensor indices attached to the vector- or tensor-valued components of such variables are systematically placed to the right of either the subscript H or T.

In conjunction with the two temperature fields θ_{H} and θ_{T}, we admit two distinct entropies η_{H} and η_{T} associated with the thermal and turbulent motions, respectively. Recalling the balance laws (2.7), $(N = 1, 2, \ldots, K)$, we identify η_{H} with the scalar density η^1; and, as in the paper of Green & Naghdi (1977*a*), introduce the thermal external rate of supply of entropy s_{H}, the thermal internal rate of supply of entropy ξ_{H}, the thermal entropy flux $\boldsymbol{p}_{\mathrm{H}}$ and identify these with corresponding quantities in (2.7) for $N = 1$. For the reader's later convenience, we record here a list of these identifications as follows:

$$\eta_{\mathrm{H}} = \eta^1, \quad s_{\mathrm{H}} = s^1, \quad \xi_{\mathrm{H}} = \xi^1, \quad \boldsymbol{p}_{\mathrm{H}} = \boldsymbol{p}^1. \tag{4.1}$$

Similarly, we identify the turbulent entropy density η_{T} with η^2 in (2.7) and associated with this admit the turbulent external rate of supply of entropy s_{T}, the turbulent internal rate of entropy ξ_{T}, the turbulent entropy flux $\boldsymbol{p}_{\mathrm{T}}$ and identify these with corresponding quantities in (2.7) for $N = 2$ as follows:

$$\eta_{\mathrm{T}} = \eta^2, \quad s_{\mathrm{T}} = s^2, \quad \xi_{\mathrm{T}} = \xi^2, \quad \boldsymbol{p}_{\mathrm{T}} = \boldsymbol{p}^2. \tag{4.2}$$

Then, the local forms of the balance laws for thermal and turbulent entropies follow from (2.11) for $N = 1, 2$ and are given by

$$\rho\dot{\eta}_{\mathrm{H}} = \rho(s_{\mathrm{H}} + \xi_{\mathrm{H}}) - \operatorname{div} \boldsymbol{p}_{\mathrm{H}}, \tag{4.3}$$

and

$$\rho\dot{\eta}_{\mathrm{T}} = \rho(s_{\mathrm{T}} + \xi_{\mathrm{T}}) - \operatorname{div} \boldsymbol{p}_{\mathrm{T}}, \tag{4.4}$$

respectively.

We assume that the rate of heat supply r and the heat flux $\boldsymbol{q}$, which occur in the energy equation $(2.10)_5$, can be expressed as the sum of their respective thermal fields r_{H}, $\boldsymbol{q}_{\mathrm{H}}$ and turbulent fields r_{T}, $\boldsymbol{q}_{\mathrm{T}}$ so that

$$r = r_{\mathrm{H}} + r_{\mathrm{T}}, \quad \boldsymbol{q} = \boldsymbol{q}_{\mathrm{H}} + \boldsymbol{q}_{\mathrm{T}}. \tag{4.5}$$

We further assume that fields s_{H}, $\boldsymbol{p}_{\mathrm{H}}$ and s_{T}, $\boldsymbol{p}_{\mathrm{T}}$ in the entropy balance laws (2.7) for $N = 1, 2$, or equivalently (4.3) and (4.4), are related to the quantities in (4.5) by

$$r_{\mathrm{H}} = \theta_{\mathrm{H}} s_{\mathrm{H}}, \quad \boldsymbol{q}_{\mathrm{H}} = \theta_{\mathrm{H}} \boldsymbol{p}_{\mathrm{H}}, \tag{4.6}$$

and

$$r_{\mathrm{T}} = \theta_{\mathrm{T}} s_{\mathrm{T}}, \quad \boldsymbol{q}_{\mathrm{T}} = \theta_{\mathrm{T}} \boldsymbol{p}_{\mathrm{T}}. \tag{4.7}$$

The definitions (4.6) and (4.7) parallel those employed in the procedure of Green & Naghdi (1977*a*, equations (2.2) and (2.5)). However, it should be noted that while r_{H}, r_{T} and $\boldsymbol{q}_{\mathrm{H}}$, $\boldsymbol{q}_{\mathrm{T}}$ are additive in accordance with $(4.5)_{1,2}$, the remaining thermal and turbulent fields in (4.3) and (4.4) are not additive and represent independent quantities.

The energy equation $(2.10)_5$ was obtained from the balance of energy $(2.6)_5$ after the elimination of the body force $\boldsymbol{b}$ and the director body force $\boldsymbol{l}$ with the help of the local equations $(2.10)_{2,3}$. A further reduced form of the energy equation can now be obtained by multiplying (4.3) by θ_{H} and (4.4) by θ_{T} and with the use of $(4.5)_{1,2}$ subtracting the result from $(2.6)_5$. This leads to the following reduced energy equation

$$\rho(\theta_{\mathrm{T}}\dot{\eta}_{\mathrm{T}} + \theta_{\mathrm{H}}\dot{\eta}_{\mathrm{H}} - \dot{\epsilon}) = \rho(\theta_{\mathrm{T}}\xi_{\mathrm{T}} + \theta_{\mathrm{H}}\xi_{\mathrm{H}}) + \boldsymbol{p}_{\mathrm{T}}\cdot\boldsymbol{g}_{\mathrm{T}} + \boldsymbol{p}_{\mathrm{H}}\cdot\boldsymbol{g}_{\mathrm{H}} - P, \tag{4.8}$$

where the mechanical power P is defined by (2.12) and the thermal and turbulent temperature gradients are defined by

$$\boldsymbol{g}_{\mathrm{H}} = \operatorname{grad}\theta_{\mathrm{H}}, \quad \boldsymbol{g}_{\mathrm{T}} = \operatorname{grad}\theta_{\mathrm{T}}, \tag{4.9}$$

respectively. In terms of a Helmholtz free energy ψ defined by

$$\psi = \epsilon - \theta_{\mathrm{H}}\eta_{\mathrm{H}} - \theta_{\mathrm{T}}\eta_{\mathrm{T}}, \tag{4.10}$$

the reduced energy equation (4.8) may be written in the alternative form

$$\rho(\dot{\psi}+\eta_H\dot{\theta}_H+\eta_T\dot{\theta}_T)+\rho(\theta_H\xi_H+\theta_T\xi_T)+\boldsymbol{p}_H\cdot\boldsymbol{g}_H+\boldsymbol{p}_T\cdot\boldsymbol{q}_T-P=0. \tag{4.11}$$

According to the procedure indicated at the end of the next section, the reduced energy equation (4.11) will be regarded as an identity for every choice of the independent variables $\boldsymbol{v}$, w, θ_H, θ_T. Then, the two balance equations (4.3) and (4.4) will be used as the equations for the determination of the temperature fields θ_H, θ_T.

5. A model of turbulent flow

A model for turbulence is adopted here which incorporates several significant observations made by Townsend (1956, 1976). (Direct quotations of Townsend's observations and related statements by other researchers are listed in Appendix A.) The model is based on the assumptions that

(1) many macroscopic features of the flow are controlled by a set of large eddies, or vortices, on the microscopic level; and

(2) the axial vorticity vectors of these eddies tend to align themselves along the particular principal direction of the rate of deformation tensor (calculated from the macroscopic velocity gradient) along which direction the associated eigenvalue (the rate of logarithmic stretch) is maximum.

The rate of deformation tensor $\boldsymbol{A}$ (with cartesian components A_{ij}) and the vorticity tensor $\boldsymbol{W}$ (with cartesian components W_{ij}) calculated from the macroscopic velocity gradient $\boldsymbol{L}$ are defined here in the usual way by

$$\left.\begin{aligned} \boldsymbol{L} &= \operatorname{grad}\boldsymbol{v} = v_{i,j}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j,\\ \boldsymbol{A} &= A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j, \quad \boldsymbol{W} = W_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j,\\ A_{ij} &= \tfrac{1}{2}(v_{i,j}+v_{j,i}), \quad W_{ij} = \tfrac{1}{2}(v_{i,j}-v_{j,i}), \end{aligned}\right\} \tag{5.1}$$

where the grad stands for the gradient operator with respect to the place $\boldsymbol{x}$ keeping t fixed and the symbol $\otimes$ denotes tensor product. (The use of the symbol $\boldsymbol{A}$ for the rate of the deformation tensor here, instead of the more customary symbol $\boldsymbol{D}$ (with cartesian components d_{ij}), is to avoid possible confusion with the notation for the reference value of the director introduced in §2 and the components $d_{i,j}$ of the director gradient in such expressions as (5.33).)

Consider now a particular turbulent flow where the velocity, on the macroscopic scale, vanishes for all times and the thermal temperature θ_H and the mass density ρ are constant. In order to properly describe the decay of the turbulent fluctuations into thermal heat, as well as other processes occurring in such a flow, an additional scalar parameter is needed in conjunction with the turbulent temperature θ_T, which is associated in some way with the typical length scale of the class of eddies containing most of the kinetic energy of the microscopic turbulent fluctuations. Examples of such a parameter are the 'mixing length' of Prandtl (1925), the 'turbulent dissipation' of Launder *et al.* (1975), and the 'pseudo-vorticity' of Saffman (1970). Rather than using any of the above variables, we introduce an 'eddy density' ρ_E representing the number of eddies of the aforementioned class per unit volume. A well-known relation in the kinetic theory of gases states that the mean free path is inversely proportional to the number of molecules per unit volume and a similar (although not identical) relation can be assumed to exist between the 'mixing length' and the eddy density ρ_E. If the use of one of the other previously mentioned variables is desired, however, the eddy density can be replaced by this other variable without difficulty.

A balance law for eddy density may be stated by assuming that the time rate of change of eddy density within a material volume in the current configuration is equal to the rate of supply of eddies within that volume, or

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho_{\mathrm{E}}\,\mathrm{d}v = \int_{\mathscr{P}} \xi_{\mathrm{E}}\,\mathrm{d}v, \tag{5.2}$$

where ξ_{E} is the rate of supply of eddies per unit volume. Then, under the usual continuity assumptions, the local form of (5.2) is

$$\dot{\rho}_{\mathrm{E}} + \rho_{\mathrm{E}} v_{i,i} = \xi_{\mathrm{E}}. \tag{5.3}$$

Clearly, when the number of eddies is conserved within a material volume, ξ_{E} vanishes and (5.3) has the same form as the local conservation of mass. It may be observed that (5.3) can be obtained as a special case of the balance law for additional scalar variables (2.7) with the identification

$$\rho_{\mathrm{E}} = \rho\eta^3, \quad \xi_{\mathrm{E}} = \rho(\xi^3 + s^3), \quad \boldsymbol{p}^3 = \mathbf{0},$$

and after using the local conservation of mass $(2.10)_1$. By not admitting a flux of eddies in equation (5.2) it is implicitly implied that the eddy centres have the same velocity as the macroscopic velocity. Although this assumption may not always be completely justified, it represents a reasonable approximation when the gradient in eddy density is gradual along the flow direction. An alternative approach is to assign a velocity to the eddy centres which is not necessarily the same as the macroscopic velocity $\boldsymbol{v}$ and express the eddy flux as proportional to the difference of these velocities. Developments similar to this, involving relative velocities, are commonly used in mixture theories with two or more constituents (see, for example, Green & Naghdi 1978). However, the latter approach unnecessarily complicates the analysis; and for the present, we adopt the simple form (5.2) for the balance of the eddy density.

For certain turbulent flows with uniform velocity it has been observed that the diagonal components of the stress tensor t_{ij} are not necessarily equal. This phenomena is commonly associated with flow 'anisotropy' in the literature (see Townsend 1976). This effect is regarded here to be caused by preferential orientation of large eddies; and, in this connection, we now admit a single director $\boldsymbol{d}$ as an additional kinematical ingredient. The director, on a macroscopic level, is associated both with the extent of alignment of the large eddies (as discussed under (2) in the opening paragraph of this section) and with the microscopic vorticity enhancement of these aligned eddies. (Such vorticity enhancement effects on the microscopic level due to continuous straining in the macroscopic flow field are commonly referred to as 'vortex stretching' in the literature.) We denote by $\boldsymbol{a}^3$ the unit principal direction of the rate of deformation tensor A_{ij} with which is associated the maximum rate of (macroscopic) logarithmic stretch $\phi^{(3)}$. It is already indicated in §2 that the director $\boldsymbol{d}$ is assumed to have the physical dimension of length. Prior to specification of additional properties of the director $\boldsymbol{d}$, it is helpful to discuss the effects of certain microscopic flow features on the macroscopic 'anisotropy.' In particular, an increase in the 'anisotropy' of the macroscopic flow is thought to be caused at the microscopic level either by vortex 'stretching', with an accompanying increase in the vorticity of the aligned eddies, or by an increase in the number of aligned eddies in a given fluid volume. Moreover, it is assumed here that the macroscopic 'anisotropy' vanishes when either the vortex 'stretching' or the number of aligned eddies vanishes. In light of this background and in the context of turbulent flow, we now specify the following additional properties for $\boldsymbol{d}$:

(i) the magnitude of $\boldsymbol{d}$ is proportional to the difference in magnitude of the microscopic vorticity vectors of the aligned and unaligned eddies times the ratio of number of large aligned eddies to total number of large eddies at a given (macroscopic) material point; and

(ii) the direction of $\boldsymbol{d}$ is identified with the principal direction $\boldsymbol{a}^3$, where (in accord with (2) of the model description) $\boldsymbol{a}^3$ has the same direction as that along which the axes of the large eddies tend to become aligned. (5.4)

To determine the desired kinematical properties associated with the rate of deformation of an arbitrary material line element, consider three mutually orthogonal material line elements which are identified as $\mathrm{d}\boldsymbol{x}_M$ in the current configuration and as $\mathrm{d}\boldsymbol{X}_M$ in the fixed reference configuration, where

$$\mathrm{d}\boldsymbol{x}_M = \boldsymbol{F}\mathrm{d}\boldsymbol{X}_M (M = 1, 2, 3), \tag{5.5}$$

and $\boldsymbol{F}$ is the deformation gradient defined in §2. Let the magnitude of $\mathrm{d}\boldsymbol{X}_M$ and $\mathrm{d}\boldsymbol{x}_M$ be denoted by $\mathrm{d}S_M$ and $\mathrm{d}s_M$, respectively, and introduce the unit vectors $\boldsymbol{M}_M$ in the direction of $\mathrm{d}\boldsymbol{X}_M$ and the unit vectors $\boldsymbol{m}_M$ in the direction of $\mathrm{d}\boldsymbol{x}_M$. In general, the line elements $\mathrm{d}\boldsymbol{X}_M$ undergo both stretch and rotation and the ratios $(\mathrm{d}s_1/\mathrm{d}S_1,\ \mathrm{d}s_2/\mathrm{d}S_2,\ \mathrm{d}s_3/\mathrm{d}S_3)$ denoted by $\lambda_M (M = 1, 2, 3)$ are called the stretch of the line elements. (The temporary notation for the unit vectors $\boldsymbol{m}_M = (\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)$ between (5.6)–(5.9) of this section need not cause confusion with the use of the symbols $\boldsymbol{m}$ and $\boldsymbol{m}_i$ for a different purpose in the conservation equations (2.6) and (2.10).) The above observations may be summarized as

$$\mathrm{d}\boldsymbol{X}_M = \boldsymbol{M}_M \mathrm{d}S_M, \quad \mathrm{d}\boldsymbol{x}_M = \boldsymbol{m}_M \mathrm{d}s_M, \quad \lambda_M = \mathrm{d}s_M/\mathrm{d}S_M \quad \text{(no sum on } M). \tag{5.6}$$

Substitution of $(5.6)_{1,2}$ into (5.5) and use of $(5.6)_3$ results in

$$\lambda_M \boldsymbol{m}_M = \boldsymbol{F}\boldsymbol{M}_M \quad \text{(no sum on } M), \tag{5.7}$$

whose material derivative after dividing by $\lambda_M (M = 1, 2, 3)$ may be expressed as

$$(\dot{\lambda}_M/\lambda_M)\,\boldsymbol{m}_M + \dot{\boldsymbol{m}}_M = \boldsymbol{L}\boldsymbol{m}_M = (\boldsymbol{A}+\boldsymbol{W})\,\boldsymbol{m}_M \quad \text{(no sum on } M), \tag{5.8}$$

where the velocity gradient $\boldsymbol{L}$ and the second order tensors $\boldsymbol{A}$ and $\boldsymbol{W}$ are defined by (5.1). After taking the scalar product of (5.8) with $\boldsymbol{m}_M$ and observing that $\dot{\boldsymbol{m}}_M \cdot \boldsymbol{m}_M = 0$ since $\boldsymbol{m}_M$ is a unit vector, we obtain

$$\dot{\lambda}_M/\lambda_M = \mathrm{d}/\mathrm{d}t\,(\ln \lambda_M) = \boldsymbol{A}\boldsymbol{m}_M \cdot \boldsymbol{m}_M \quad \text{(no sum on } M), \tag{5.9}$$

where the left-hand side of (5.9) represents the rate of logarithmic stretch.

Because $\boldsymbol{A}$ is a real-valued symmetric tensor, it possesses three mutually orthogonal and linearly independent unit principal directions $\boldsymbol{a}^i$ and associated principal values $\phi^{(i)}$, $(i = 1, 2, 3)$ such that

$$\boldsymbol{A}\boldsymbol{a}^i = \phi^{(i)}\boldsymbol{a}^i, \quad \boldsymbol{a}^i \cdot \boldsymbol{a}^j = \delta_{ij}, \tag{5.10}$$

where δ_{ij} stands for Kronecker delta and the parentheses around an index signifies suspension of the summation convention. (Although all vectors and tensors in this paper are referred to rectangular cartesian coordinates with orthonormal basis $\boldsymbol{e}_i$, as in (5.4) and (5.10), for convenience we consistently employ superscripts to distinguish between the unit principal directions $(\boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{a}^3)$ and associated principal values $(\phi^{(1)}, \phi^{(2)}, \phi^{(3)})$.) The scalar product of $(5.10)_1$ with each of $\boldsymbol{a}^i$, after using $(5.10)_2$, gives

$$\phi^{(i)} = \boldsymbol{A}\boldsymbol{a}^i \cdot \boldsymbol{a}^i \quad \text{(no sum on } i\text{)}. \tag{5.11}$$

Identifying the arbitrary unit vector $\boldsymbol{m}_i$ with the unit principal direction $\boldsymbol{a}^i$, we find from (5.8), (5.9), and (5.11) that

$$\phi^{(i)} = \frac{\dot{\lambda}_i}{\lambda_i} \equiv \left(\frac{\dot{\lambda}}{\lambda}\right)^{(i)} \quad \text{(no sum on } i\text{)} \tag{5.12}$$

and

$$\dot{\boldsymbol{a}}^i = \boldsymbol{L}\boldsymbol{a}^i - \left(\frac{\dot{\lambda}}{\lambda}\right)^{(i)} \boldsymbol{a}^i \quad \text{(no sum on } i\text{)}. \tag{5.13}$$

Let $\boldsymbol{\omega}$ denote the axial vector associated with the vorticity tensor $\boldsymbol{W}$ so that, for any vector $\boldsymbol{V}$, $\boldsymbol{W}\boldsymbol{V} = \boldsymbol{\omega} \times \boldsymbol{V}$. Then, with the use of $(5.10)_1$ and (5.12), it follows from (5.8) that

$$\dot{\boldsymbol{a}}^i = \boldsymbol{W}\boldsymbol{a}^i = \boldsymbol{\omega} \times \boldsymbol{a}^i, \tag{5.14}$$

i.e. the axial vector $\boldsymbol{\omega}$ is the angular velocity of line elements which are parallel to the principal directions $\boldsymbol{a}^i$.

We now recall the property (ii) listed in (5.4). Because $\boldsymbol{d}$ is parallel to the unit principal direction $\boldsymbol{a}^3$ and because $\boldsymbol{a}^3$ is orthogonal to the unit principal directions $\boldsymbol{a}^\alpha$, $(\alpha = 1, 2)$, the constraint on the director alignment can be stated as

$$\boldsymbol{d} = d\boldsymbol{a}^3, \quad \boldsymbol{a}^\alpha \cdot \boldsymbol{d} = 0 \quad (\alpha = 1, 2). \tag{5.15}$$

Differentiating $(5.15)_2$ with respect to time gives

$$\dot{\boldsymbol{a}}^\alpha \cdot \boldsymbol{d} + \boldsymbol{a}^\alpha \cdot \boldsymbol{w} = 0. \tag{5.16}$$

From (5.13) and (5.15) $\dot{\boldsymbol{a}}^\alpha \cdot \boldsymbol{d} = \boldsymbol{L}\boldsymbol{a}^\alpha \cdot \boldsymbol{d}$, so that (5.16) can be written as

$$\boldsymbol{d} \cdot \boldsymbol{L}\boldsymbol{a}^\alpha + \boldsymbol{a}^\alpha \cdot \boldsymbol{w} = 0 \tag{5.17}$$

or in component form as

$$d_i a_j^\alpha v_{i,j} + a_i^\alpha w_i = 0 \quad (\alpha = 1, 2), \tag{5.18}$$

where

$$\boldsymbol{a}^i = a_j^i \boldsymbol{e}_j. \tag{5.19}$$

(A discussion of constraints for a directed medium with a single director is included in Green *et al.* (1970, §6). The development between (5.15) and (5.23) is analogous to that of a similar constrained theory carried out in a different context by Naghdi (1982, §6.2), where additional references on the subject can be found.)

For the constrained theory under discussion, we assume that each of the response functions t_{ij}, m_{ij}, and k_i are determined to within an additive constraint response $\bar{t}_{ij}$, $\bar{m}_{ij}$, and $\bar{k}_i$ so that

$$t_{ij} = \bar{t}_{ij} + \hat{t}_{ij}, \quad m_{ij} = \bar{m}_{ij} + \hat{m}_{ij}, \quad k_i = \bar{k}_i + \hat{k}_i, \tag{5.20}$$

where $\hat{t}_{ij}$, $\hat{m}_{ij}$ and $\hat{k}_i$ are to be specified by constitutive equations and the constraint responses which are *workless* are independent of the kinematical variables $(v_{i,j}, w_i, w_{i,j})$ and are only arbitrary functions of $\boldsymbol{x}, t$. Thus, recalling the expression (2.12) for mechanical power, we have

$$\bar{t}_{ij} v_{i,j} + \bar{k}_i w_i + \bar{m}_{ij} w_{i,j} = 0. \tag{5.21}$$

Multiplying the two director alignment constraints in (5.18) by the Lagrange multipliers β^α and subtracting from (5.21) gives

$$(\bar{t}_{ij}-\sum_\alpha \beta^\alpha d_i a^\alpha_j)\, v_{i,j}+(\bar{k}_i-\sum_\alpha \beta^\alpha a^\alpha_i)\, w_i+\bar{m}_{ij}\, w_{i,j}=0, \quad (\alpha=1,2). \tag{5.22}$$

From (5.22) and the fact that $\bar{t}_{ij}$, $\bar{k}_i$ and $\bar{m}_{ij}$ are independent of $v_{i,j}$, w_i, and $w_{i,j}$, it follows that

$$\bar{t}_{ij}=\sum_\alpha \beta^\alpha d_i a^\alpha_j, \quad \bar{k}_i=\sum_\alpha \beta^\alpha a^\alpha_i, \quad \bar{m}_{ij}=0 \quad (\alpha=1,2). \tag{5.23}$$

It can be easily verified that the constraint responses (5.23) satisfy the moment of momentum equation $(2.10)_4$ identically.

The ordinary and director momentum equations $(2.10)_{2,3}$ with the use of (5.23) can be written as

$$\rho(\dot{v}_i+y_1\dot{w}_i)=\rho b_i+\hat{t}_{ij,j}+\sum_\alpha (\beta^\alpha d_i a^\alpha_j)_{,j}, \tag{5.24}$$

$$\rho(y_1\dot{v}_i+y_2\dot{w}_i)=\rho l_i-\hat{k}_i+\hat{m}_{ij,j}-\sum_\alpha \beta^\alpha a^\alpha_i. \tag{5.25}$$

It is convenient to introduce here the temporary abbreviations

$$\hat{b}_i=b_i-(\dot{v}_i+y_1\dot{w}_i), \quad \hat{l}_i=l_i-(y_1\dot{v}_i+y_2\dot{w}_i). \tag{5.26}$$

Then, after multiplying (5.25) by d_i and using $(5.10)_2$ and $(5.15)_2$, we have

$$\rho\hat{l}_i d_i-\hat{k}_i d_i+d_i\hat{m}_{ij,j}=0, \tag{5.27}$$

where we have also used $(5.26)_2$. Combining (5.24) and (5.25) and using (5.26) we obtain

$$0=\rho\hat{b}_i+\frac{\partial}{\partial x_j}\{\hat{t}_{ij}+d_i[-\hat{k}_j+\rho\hat{l}_j+\hat{m}_{jl,l}]\}. \tag{5.28}$$

Because $d_i=da^3_i$ by $(5.15)_1$, it is clear that the system of equations (5.27) and (5.28) represent a set of four equations for the determination of four primary unknowns d and v_i.

For later reference, we introduce here alternative kinematical variables h_i and h_{ij} defined by

$$h_i=w_i-W_{ik}d_k, \quad h_{ij}=w_{i,j}-W_{ik}d_{k,j}, \tag{5.29}$$

both of which can be easily shown to be properly invariant under superposed rigid body motions. Also, with the help of $(5.29)_2$, the components of the material time derivative of the director gradient can be expressed as

$$\overline{\dot{d_{i,j}}}=h_{ij}+W_{ik}d_{k,j}-v_{k,j}d_{i,k}. \tag{5.30}$$

As promised in §2, we now describe a procedure for the utilization of the balance equations (2.10) and (2.11) and the constitutive equations for the turbulent viscous fluids to be introduced in §6. In this connection, we note that the local conservation laws (2.10), the thermal and turbulent entropy balance laws (4.3) and (4.4) and the balance of eddies (5.3) involve a set of 19 functions. These consist of the two velocities $\boldsymbol{v}$ and $\boldsymbol{w}$ defined by $(2.2)_{1,2}$, or the two functions χ and $\mathscr{D}$ in $(2.1)_{1,2}$, and the two temperature-like variables θ_{H} and θ_{T}

(introduced in §4) as the independent variables; and the various thermomechanical fields†, namely

$$\{t_{ji}, m_{ji}, k_i\} \quad \text{and} \quad \{\psi, \eta_{\mathrm{H}}, \eta_{\mathrm{T}}, p_{\mathrm{H}i}, p_{\mathrm{T}i}, \xi_{\mathrm{H}}, \xi_{\mathrm{T}}, \xi_E\}, \tag{5.31}$$

as well as

$$\{b_i, l_i, s_{\mathrm{H}}, s_{\mathrm{T}}\}. \tag{5.32}$$

We assume that the fields (5.31) are specified by constitutive equations which may depend on the kinematical variables χ, $\mathscr{D}$ and the temperatures θ_{H} and θ_{T}, their space and time derivatives, as well as the whole history of these variables. Then, following Green & Naghdi (1977*a*), we adopt the following procedure in using the balance laws:

(1) the field equations are assumed to hold for an arbitrary choice of the functions $\boldsymbol{v}$, $\boldsymbol{w}$ (or χ, $\mathscr{D}$), θ_{H} and θ_{T} including, if required, any arbitrary choice of the space and time derivatives of these functions;

(2) the fields (5.31) are calculated from their respective constitutive equations;

(3) the values of the fields (5.32) can then be found from the balances of linear and director momentum $(2.10)_{2,3}$ and the entropy balance laws (4.3) and (4.4); and

(4) the equation resulting from the balance of moment of momentum $(2.10)_4$ and the reduced energy equation (4.11) (resulting from the balance of energy), will be regarded as identities for every choice of the kinematical variables (2.1), or the corresponding velocities (2.2), and the temperatures θ_{H}, θ_{T} identified in §4. These equations will then place restrictions on the constitutive equations. Also, as should be evident from the development in §4, the thermal and turbulent entropy balance laws (4.3) and (4.4) will be used as the equations for the determination of the temperature fields θ_{H}, θ_{T}.

It should be clear that with the help of $(2.9)_{1,2}$ the component forms of the equations of motion $(2.10)_{2,3,4}$ can be easily expressed in terms of the stress tensor t_{ji}, the director stress m_{ji} and the components of other fields (referred to the basis $\boldsymbol{e}_i$) in these equations. Moreover, in view of the procedure discussed in the preceding paragraph (see item (4)) it follows from the local conservation of moment of momentum $(2.10)_4$ that the skew-symmetric part of the Cauchy stress tensor $t_{[ji]}$ is given by

$$t_{[ji]} = \tfrac{1}{2}(d_{j,k}\, m_{ik} - d_{i,k}\, m_{jk} + d_j\, k_i - d_i\, k_j), \tag{5.33}$$

and we only need to require constitutive equations for the symmetric part $t_{(ji)}$ of the Cauchy stress tensor in (5.31).

6. Constitutive equations for turbulent viscous fluids

The determinate parts of the constitutive responses in the constrained theory of §5, namely

$$\hat{t}_{(ij)}, \hat{k}_i, \hat{m}_{ij}, \tag{6.1}$$

as well as the thermal and turbulent constitutive responses

$$\psi, \eta_{\mathrm{H}}, \eta_{\mathrm{T}}, \xi_{\mathrm{H}}, \xi_{\mathrm{T}}, p_{\mathrm{H}}, p_{\mathrm{T}}, \xi_{\mathrm{E}}, \tag{6.2}$$

are assumed to depend on the current values of the set of independent variables

$$\mathscr{V}_0 = (\rho, \rho_{\mathrm{E}}, \theta_{\mathrm{H}}, \theta_{\mathrm{T}}, d_i, d_{i,j}), \tag{6.3}$$

† The mass density ρ is not included in (5.31) because it can be calculated from $(2.10)_1$. Similarly, the eddy density ρ_{E} introduced in (5.2) need not be included in the list (5.31).

as well as the rate quantities

$$\mathscr{V}_1 = (A_{ij}, h_i) \tag{6.4}$$

and the thermal and turbulent temperature gradients and turbulent temperature rate

$$\mathscr{V}_2 = (g_{Hi}, g_{Ti}, \dot{\theta}_T), \tag{6.5}$$

where A_{ij} and h_i are defined by $(5.1)_4$ and $(5.29)_1$. As will become apparent later in this section, the inclusion of $\dot{\theta}_T$ in the set of independent variables (6.5) is necessary to properly describe the rate of eddy supply ξ_E per unit volume. Thus for example, the constitutive assumption of the specific Helmholtz free energy reads as

$$\psi = \tilde{\psi}(\mathscr{V}_0, \mathscr{V}_1, \mathscr{V}_2), \tag{6.6}$$

where for clarity in the immediate discussion that follows (between (6.6) and (6.9)) we have temporarily distinguished between a function and its value by writing $\tilde{\psi}$ (instead of just ψ) for the response function on the right-hand side of (6.6). Statements similar to (6.6) hold for all other variables in (6.1) and (6.2).

It should be recalled here that the reduced energy equation (4.11) has been obtained from the energy equation $(2.6)_5$ after the elimination of the external body force $\boldsymbol{b}$ (in the form $(2.10)_5$) and the external supplies of entropy through the definitions $(4.6)_1$ and $(4.7)_1$. Further, in accordance with the procedure of Green & Naghdi (1977), the reduced energy equation (4.11) is to be regarded as an identity for *all* processes provided the external body force $\boldsymbol{b}$, the external director body force $\boldsymbol{l}$ and the external supplies of entropy s_H and s_T are chosen so as to satisfy the balance of linear momentum $(2.10)_2$, the balance of director momentum $(2.10)_3$ and the entropy balance equations (4.3) and (4.4). This procedure of restrictions on the constitutive assumption regarding the reduced energy equation (4.11) as an identity places restrictions on the original constitutive equations; and, as will become evident presently, reduces the number of response functions. In addition, all constitutive equations must remain invariant under arbitrary superposed rigid body motions. Given appropriate constitutive equations, the complete solution of a turbulent boundary value problem must satisfy the consequences of the conservation of mass $(2.10)_1$, the conservation of the linear and director momentum $(2.10)_{2,3}$, the thermal and turbulent entropy balance laws (4.3) and (4.4), the eddy density balance law (5.3), and the director alignment constraint (5.15). As noted in §2, the inertia coefficients y_1 and y_2 also require constitutive equations.

The material time derivative of the Helmholtz free energy ψ is

$$\begin{aligned}\dot{\psi} = {} & \frac{\partial\tilde{\psi}}{\partial A_{ij}}\dot{A}_{ij} + \frac{\partial\tilde{\psi}}{\partial h_i}\dot{h}_i + \frac{\partial\tilde{\psi}}{\partial d_i}h_i + \frac{\partial\tilde{\psi}}{\partial d_{i,j}}h_{ij} - \frac{\partial\tilde{\psi}}{\partial d_{k,j}}d_{k,i}A_{ij} \\ & + \left(\frac{\partial\tilde{\psi}}{\partial d_i}d_j + \frac{\partial\tilde{\psi}}{\partial d_{i,k}}d_{j,k} + \frac{\partial\tilde{\psi}}{\partial d_{k,i}}d_{k,j}\right)W_{ij} + \frac{\partial\tilde{\psi}}{\partial \rho}\dot{\rho} + \frac{\partial\tilde{\psi}}{\partial \rho_E}\dot{\rho}_E \\ & + \frac{\partial\tilde{\psi}}{\partial \theta_H}\dot{\theta}_H + \frac{\partial\tilde{\psi}}{\partial \theta_T}\dot{\theta}_T + \frac{\partial\tilde{\psi}}{\partial \dot{\theta}_T}\ddot{\theta}_T + \frac{\partial\tilde{\psi}}{\partial g_{Ti}}\dot{g}_{Ti} + \frac{\partial\tilde{\psi}}{\partial g_{Hi}}\dot{g}_{Hi},\end{aligned} \tag{6.7}$$

where we have also used $(5.29)_{1,2}$ and (5.30). Substituting (6.7) into the reduced energy equation (4.11) and using the mass conservation equation $(2.10)_1$ and the eddy balance equation (5.3) to eliminate $\dot{\rho}$ and $\dot{\rho}_E$ we obtain

$$\begin{aligned}
&\rho\frac{\partial\tilde{\psi}}{\partial A_{ij}}\dot{A}_{ij}+\rho\frac{\partial\tilde{\psi}}{\partial h_i}\dot{h}_i+\left(\rho\frac{\partial\tilde{\psi}}{\partial d_i}-\hat{k}_i\right)h_i+\left(\rho\frac{\partial\tilde{\psi}}{\partial d_{i,j}}-\hat{m}_{ij}\right)h_{ij}\\
&\quad-\left[\left(\rho\rho_E\frac{\partial\tilde{\psi}}{\partial\rho_E}+\rho^2\frac{\partial\tilde{\psi}}{\partial\rho}\right)\delta_{ij}+\rho\frac{\partial\tilde{\psi}}{\partial d_{k,j}}d_{k,i}+\hat{t}_{ij}\right]A_{ij}\\
&\quad+\rho\left(\frac{\partial\tilde{\psi}}{\partial d_i}d_j+\frac{\partial\tilde{\psi}}{\partial d_{i,k}}d_{j,k}+\frac{\partial\tilde{\psi}}{\partial d_{k,i}}d_{k,j}\right)W_{ij}\\
&\quad+\rho\left(\frac{\partial\tilde{\psi}}{\partial\theta_H}+\eta_H\right)\dot{\theta}_H+\rho\left(\frac{\partial\tilde{\psi}}{\partial\theta_T}+\eta_T\right)\dot{\theta}_T+\rho\frac{\partial\tilde{\psi}}{\partial\dot{\theta}_T}\ddot{\theta}_T+\rho\frac{\partial\tilde{\psi}}{\partial g_{Hi}}\dot{g}_{Hi}\\
&\quad+\rho\frac{\partial\tilde{\psi}}{\partial g_{Ti}}\dot{g}_{Ti}+\rho\frac{\partial\tilde{\psi}}{\partial\rho_E}\xi_E+\rho\theta_H\xi_H+\rho\theta_T\xi_T+p_{Hi}g_{Hi}\\
&\quad+p_{Ti}g_{Ti}=0.
\end{aligned}\tag{6.8}$$

Because (6.8) is linear in the variables $\dot{A}_{ij}$, $\dot{h}_i$, h_{ij}, W_{ij}, $\dot{\theta}_H$, $\ddot{\theta}_T$, $\dot{g}_{Hi}$, $\dot{g}_{Ti}$, with coefficients which are independent of these variables, we may conclude that

$$\frac{\partial\tilde{\psi}}{\partial A_{ij}}=\frac{\partial\tilde{\psi}}{\partial h_i}=\frac{\partial\tilde{\psi}}{\partial g_{Ti}}=\frac{\partial\tilde{\psi}}{\partial g_{Hi}}=\frac{\partial\tilde{\psi}}{\partial\dot{\theta}_T}=0$$

and hence

$$\psi=\psi(\mathscr{V}_0),\tag{6.9}$$

where ψ on the right-hand side of (6.9) is now a different function from that in (6.6). Given the result (6.9), we further conclude from (6.8) that

$$\left.\begin{aligned}
&\eta_H+\frac{\partial\psi}{\partial\theta_H}=0,\quad \hat{m}_{ij}=\rho\frac{\partial\psi}{\partial d_{i,j}},\\
&\frac{\partial\psi}{\partial d_i}d_j+\frac{\partial\psi}{\partial d_{i,k}}d_{j,k}+\frac{\partial\psi}{\partial d_{k,i}}d_{k,j}=\frac{\partial\psi}{\partial d_j}d_i+\frac{\partial\psi}{\partial d_{j,k}}d_{i,k}+\frac{\partial\psi}{\partial d_{k,j}}d_{k,i},
\end{aligned}\right\}\tag{6.10}$$

and

$$\begin{aligned}
&\left(\rho\frac{\partial\psi}{\partial d_i}-\hat{k}_i\right)h_i-\left[\left(\rho\rho_E\frac{\partial\psi}{\partial\rho_E}+\rho^2\frac{\partial\psi}{\partial\rho}\right)\delta_{ij}+\rho\frac{\partial\psi}{\partial d_{k,j}}d_{k,i}+\hat{t}_{ij}\right]A_{ij}\\
&\qquad+\rho\left(\frac{\partial\psi}{\partial\theta_T}+\eta_T\right)\dot{\theta}_T+\rho\theta_H\xi_H+\rho\theta_T\xi_T+\rho\frac{\partial\psi}{\partial\rho_E}\xi_E+p_{Hi}g_{Hi}+p_{Ti}g_{Ti}=0.
\end{aligned}\tag{6.11}$$

To avoid unnecessary complications in the discussion of the constitutive equations, in the rest of this section we specialize further the constitutive assumptions introduced earlier (in the paragraph containing (6.6)). Keeping in mind the constitutive results (6.9) and (6.10), we consider a special case in which the various response functions are either linear or quadratic in the rate variables (6.4) and possibly also linear or quadratic in the variables (6.5), but with coefficients which may depend on the set of variables (6.3). Given the constitutive results $(6.10)_{1,2}$, we now assume that:

$$\left.\begin{array}{l} \hat{t}_{(ij)} \text{ and } \hat{k}_i \text{ are linear in (6.4);} \\ p_{\mathrm{H}i} \text{ is linear in } g_{\mathrm{H}i} \text{ and vanishes when } g_{\mathrm{H}i} = 0; \\ p_{\mathrm{T}i} \text{ is linear in } g_{\mathrm{T}i} \text{ and vanishes when } g_{\mathrm{T}i} = 0; \\ \xi_{\mathrm{H}} \text{ is quadratic in (6.4) and } g_{\mathrm{H}i}; \\ \xi_{\mathrm{T}} \text{ is quadratic in (6.4) and } g_{\mathrm{T}i}; \\ \xi_{\mathrm{E}} \text{ is linear in (6.4) and } \dot{\theta}_{\mathrm{T}}; \text{ and} \\ \eta_{\mathrm{T}} \text{ is independent of the variables (6.4) and (6.5).} \end{array}\right\} \tag{6.12}$$

It should be emphasized that the constitutive assumptions $(6.12)_{4-7}$ are compatible with the remaining assumptions in (6.12) and that at this stage in our development the various coefficients for the responses (6.12) depend on the variables (6.3). Moreover, these response functions must be such that (i) the reduced energy equation (6.11) is satisfied identically for all motions and (ii) the forms of the constitutive coefficients in the various constitutive equations properly satisfy the invariance requirements under superposed rigid body motions.

We now introduce further simplifying assumptions in that all response coefficients in the constitutive assumptions (6.12), except those for η_{T}, are quadratic in d_i and that those coefficients which are not multiplied by one or more of the rate quantities (6.4) or the variables (6.5) are also linear in the director gradient $d_{i,j}$. In the light of all previous assumptions and consistent with the use of the reduced energy equation (6.11) as an identity for all thermomechanical processes, the response coefficients for η_{T} must necessarily be linear in $d_{i,j}$ and quartic in d_i (in the form $d_i d_i d_j d_j$). With these additional assumptions, the various response coefficients depend only on the set of variables

$$\mathscr{U} = (\rho, \rho_{\mathrm{E}}, \theta_{\mathrm{H}}, \theta_{\mathrm{T}}), \tag{6.13}$$

instead of (6.3). With the use of invariance requirements under superposed rigid body motions and after observing that all quantities in (6.1)–(6.5) are objective, it can be shown that the various coefficients in (6.12) and the free energy ψ must be hemitropic functions of their arguments. For our present purpose, however, it will suffice to require that ψ and these coefficients are only isotropic functions. With the latter simplification, the final forms of the constitutive results are as follows:

$$\psi = \psi_0 + \psi_1 d_{i,i} + \psi_2 d_i d_i, \tag{6.14}$$

$$\eta_{\mathrm{T}} = -\frac{\partial \psi}{\partial \theta_{\mathrm{T}}} - \frac{\partial \psi}{\partial \rho_{\mathrm{E}}} (\gamma_0 + \gamma_1 d_i d_i), \tag{6.15}$$

$$\hat{k}_i = \nu_0 d_i + \nu_1 h_i + \nu_2 d_l d_l h_i + \nu_3 d_i d_j h_j + 2\nu_4 d_l A_{il} + \nu_5 d_i A_{ll}, \tag{6.16}$$

$$\begin{aligned} \hat{t}_{(ij)} = {} & \mu_0 \delta_{ij} + \mu_1 \delta_{ij} d_{l,l} + \mu_2 (d_{i,j} + d_{j,i}) + \mu_3 \delta_{ij} d_l d_l + \mu_4 d_i d_j \\ & + \mu_5 \delta_{ij} d_l h_l + \mu_6 (d_i h_j + d_j h_i) \\ & + \mu_7 \delta_{ij} A_{ll} + 2\mu_8 A_{ij} + \mu_9 \delta_{ij} A_{ll} d_m d_m + \mu_{10} \delta_{ij} d_l d_m A_{lm} \\ & + \mu_{11} d_i d_j A_{ll} + 2\mu_{12} d_l d_l A_{ij} + 2\mu_{13} d_m (d_j A_{im} + d_i A_{jm}), \end{aligned} \tag{6.17}$$

$$\hat{t}_{[ij]} = \tfrac{1}{2}\rho \left(\frac{\partial \psi}{\partial d_{j,k}} d_{i,k} - \frac{\partial \psi}{\partial d_{i,k}} d_{j,k} \right) + \tfrac{1}{2}(\nu_1 + \nu_2 d_l d_l)(d_i h_j - d_j h_i) + \nu_4 d_l (d_i A_{jl} - d_j A_{il}), \tag{6.18}$$

$$p_{\mathrm{H}i} = -(\kappa_0 + \kappa_1 d_l d_l)\, g_{\mathrm{H}i}, \tag{6.19}$$

$$p_{\mathrm{T}i} = -(\lambda_0 + \lambda_1 d_l d_l)\, g_{\mathrm{T}i}, \tag{6.20}$$

$$\xi_{\mathrm{E}} = (\gamma_0 + \gamma_1 d_i d_i)\,\dot{\theta}_{\mathrm{T}} + \gamma_2 d_i h_i + \gamma_3 A_{ii} + \gamma_4 d_l d_l A_{ii} + \gamma_5 d_i d_j A_{ij}, \tag{6.21}$$

$$\begin{aligned} \xi_{\mathrm{H}} = {} & \phi_0 + \phi_1 d_{i,i} + \phi_2 d_i d_i + \frac{1}{\rho\theta_{\mathrm{H}}}(\kappa_0 + \kappa_1 d_l d_l)\, g_{\mathrm{H}i} g_{\mathrm{H}i} \\ & + \phi_3 d_i h_i + (\phi_7 + \phi_8 d_l d_l) A_{ii} + \phi_9 d_i d_j A_{ij} + (\phi_{10} + \phi_{11} d_l d_l) h_i h_i \\ & + \phi_{12} d_i d_j h_i h_j + \phi_{13} d_i h_i A_{ll} + 2\phi_{14} d_l h_i A_{il} \\ & + (\phi_{15} + \phi_{16} d_m d_m) A_{ii} A_{ll} + 2(\phi_{17} + \phi_{18} d_l d_l) A_{ij} A_{ij} + 2\phi_{19} d_i d_j A_{ij} A_{ll} \\ & + 4\phi_{20} d_i d_j A_{il} A_{jl}, \end{aligned} \tag{6.22}$$

and

$$\begin{aligned} \xi_{\mathrm{T}} = {} & -\frac{\theta_{\mathrm{H}}}{\theta_{\mathrm{T}}}(\phi_0 + \phi_1 d_{i,i} + \phi_2 d_i d_i) + \frac{1}{\rho\theta_{\mathrm{T}}}[\lambda_0 + \lambda_1 d_l d_l]\, g_{\mathrm{T}i} g_{\mathrm{T}i} \\ & - \frac{1}{\theta_{\mathrm{T}}}\left[\frac{\partial\psi}{\partial d_i} + \left(\theta_{\mathrm{H}}\phi_3 + \frac{\partial\psi}{\partial\rho_{\mathrm{E}}}\gamma_2 - \frac{\nu_0}{\rho}\right) d_i\right] h_i \\ & + \frac{1}{\theta_{\mathrm{T}}}\left[(\rho_{\mathrm{E}} - \gamma_3 - \gamma_4 d_l d_l)\frac{\partial\psi}{\partial\rho_{\mathrm{E}}} + \rho\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho}(\mu_0 + \mu_1 d_{l,l} + \mu_3 d_l d_l) \right. \\ & \left. - \theta_{\mathrm{H}}(\phi_7 + \phi_8 d_l d_l)\right] A_{ii} \\ & + \frac{1}{\theta_{\mathrm{T}}}\left[\frac{\partial\psi}{\partial d_{k,j}} d_{k,i} + \frac{1}{\rho}(2\mu_2 d_{i,j} + \mu_4 d_i d_j) - \gamma_5\frac{\partial\psi}{\partial\rho_{\mathrm{E}}} d_i d_j - \theta_{\mathrm{H}}\phi_9 d_i d_j\right] A_{ij} \\ & + \frac{1}{\theta_{\mathrm{T}}}\left[\frac{1}{\rho}(\nu_1 + \nu_2 d_l d_l) - \theta_{\mathrm{H}}(\phi_{10} + \phi_{11} d_l d_l)\right] h_i h_i \\ & + \frac{1}{\phi_{\mathrm{T}}}\left[\frac{\nu_3}{\rho} - \theta_{\mathrm{H}}\phi_{12}\right] d_i d_j h_i h_j + \frac{1}{\theta_{\mathrm{T}}}\left[\frac{1}{\rho}(\nu_5 + \mu_5) - \theta_{\mathrm{H}}\phi_{13}\right] d_i h_i A_{ll} \\ & + \frac{2}{\theta_{\mathrm{T}}}\left[\frac{1}{\rho}(\nu_4 + \mu_6) - \theta_{\mathrm{H}}\phi_{14}\right] d_l h_i A_{il} \\ & + \frac{1}{\theta_{\mathrm{T}}}\left[\frac{1}{\rho}(\mu_7 + \mu_9 d_l d_l) - \theta_{\mathrm{H}}(\phi_{15} + \phi_{16} d_l d_l)\right] A_{ii} A_{jj} \\ & + \frac{2}{\theta_{\mathrm{T}}}\left[\frac{1}{\rho}(\mu_8 + \mu_{12} d_l d_l) - \theta_{\mathrm{H}}(\phi_{17} + \phi_{18} d_l d_l)\right] A_{ij} A_{ij} \\ & + \frac{1}{\theta_{\mathrm{T}}}\left[\frac{1}{\rho}(\mu_{10} + \mu_{11}) - 2\theta_{\mathrm{H}}\phi_{19}\right] d_i d_j A_{ij} A_{ll} \\ & + \frac{4}{\theta_{\mathrm{T}}}\left[\frac{\mu_{13}}{\rho} - \theta_{\mathrm{H}}\phi_{20}\right] d_i d_j A_{il} A_{jl}. \end{aligned} \tag{6.23}$$

The coefficients in (6.14)–(6.23), which represent response (or material) coefficients, depend only on the variables (6.13). Any subset of these coefficients may be taken to be zero without violating the restrictions imposed by the satisfaction of the moment of momentum equation, the reduced energy equation or the invariance requirements under superposed rigid body motions. The expression (6.15) follows from the fact that, given the assumptions (6.12), the reduced energy equation (6.11) holds for arbitrary values of $\dot{\theta}_{\mathrm{T}}$. It may be observed that when the second term on the right-hand side of (6.15) vanishes, then η_{T} will have the same form as the more standard result $(6.10)_1$. Further simplification of (6.14)–(6.23) to the case of (rate-independent) inviscid flow, which would be appropriate for sufficiently intense turbulence, is discussed in §8. It should also be noted that in the absence of turbulence ($\rho_{\mathrm{E}} = \theta_{\mathrm{T}} = 0$,

$\boldsymbol{d} = \boldsymbol{o}$) and apart from the presence of thermal effects (involving the temperature θ_{H}), a number of constitutive equations such as those for $\boldsymbol{p}_{\mathrm{T}}$, ξ_{T}, ξ_{E} and the expression for the anti-symmetric stress tensor $t_{[ij]}$ vanish identically; and, then the remaining constitutive equations together with appropriate balance laws in (2.10) can be combined to yield the standard Navier–Stokes equations.

A number of important aspects of the constitutive results (6.14)–(6.23) require separate extensive studies and will be dealt with in the future. These studies pertain to possible additional restrictions on the response coefficients, which may arise from appropriate statements on the Second Law of Thermodynamics or some modification of this law appropriate to turbulent flow, as well as the identification of the various response coefficients (beyond those of ordinary viscosity or conductivity coefficients) in the turbulent regime.

In the case of an incompressible fluid, we need to consider the additional constraint

$$\delta_{ij} A_{ij} = 0. \tag{6.24}$$

The stress tensor t_{ij} can then be decomposed as

$$t_{ij} = \bar{t}_{ij} + t^*_{ij} + \tilde{t}_{ij}, \tag{6.25}$$

where as in $(5.20)_1$ the constraint response $\bar{t}_{ij}$ is caused by the director alignment constraint (5.15), t^*_{ij} is the response arising from the incompressibility constraint (6.24), and now $\tilde{t}_{ij}$ stands for the determinate part of the response which requires constitutive equations. The form $(5.23)_1$ for $\bar{t}_{ij}$ remains unchanged for an incompressible material and by a standard argument we have

$$t^*_{ij} = -p\delta_{ij}, \tag{6.26}$$

where $p = p(\boldsymbol{x}, t)$ is a Lagrange multiplier.

As to be expected, some of the constitutive equations for incompressible turbulent viscous fluids assume simpler structure relative to those in (6.14)–(6.23). Before indicating the nature of this simple structure, we observe that the constraint of incompressibility (6.24) does not affect several of the equations obtained for the compressible case, namely equations (6.14), (6.15) and (6.18)–(6.20). Again because of (6.24), a number of terms in the remaining equations which contain the scalar A_{ii} vanish identically. Moreover, in the expression for the stress tensor certain scalar multiples of δ_{ij} such as

$$\delta_{ij} d_{l,l}, \quad \delta_{ij} d_l d_l,$$

can now be absorbed into the Lagrange multiplier p, which is an arbitrary function of position and time. In the interest of brevity, we do not record here the complete system of constitutive equations for the incompressible fluid, but note that those which are affected by the incompressibility condition (6.24) may also be obtained by suppressing the effects of terms involving the following coefficients in (6.14)–(6.23):

$$\left.\begin{array}{l} \mu_0, \mu_1, \mu_3, \mu_5, \mu_7, \mu_9, \mu_{10}, \mu_{11}, \\ \gamma_3, \gamma_4, \nu_5, \\ \phi_7, \phi_8, \phi_{13}, \phi_{15}, \phi_{16}, \phi_{19}. \end{array}\right\} \tag{6.27}$$

In particular, for the incompressible fluid the expression corresponding to (6.17) of the compressible case is

$$\begin{aligned} -p\delta_{ij} + \tilde{t}_{(ij)} = -p\delta_{ij} + \mu_2(d_{i,j} + d_{j,i}) + \mu_4 d_i d_j + \mu_6(d_i h_j + d_j h_i) \\ + 2\mu_8 A_{ij} + 2\mu_{12} d_l d_l A_{ij} + 2\mu_{13} d_m (d_j A_{im} + d_i A_{jm}). \end{aligned} \tag{6.28}$$

It should be noted that of the two equations of motion (5.27) and (5.28), the first remains intact for the incompressible fluid but in the second $\hat{t}_{(ij)}$ must be replaced by the expression (6.28).

There is a school of turbulence theory, which purports the notion that t_{ij} must remain symmetric for turbulent flow. This claim is usually based upon an estimate of the 'mean' stress by integrating certain terms in the Navier-Stokes equations over appropriately defined volumes or time intervals. This estimate of 'mean' stress is then identified with the macroscopic stress. The symmetry properties of the stress tensor thus obtained depends upon the form of the weighting function used in this integration. Usually this weighting function is assumed to be one and the resulting stress tensor is then symmetric, as in Tennekes & Lumley (1972, p. 32). Such a choice may give reasonable approximate results in many instances, but it cannot be a basis of general validity due to the previously mentioned differences between the macroscopic flow, as defined here, and the standard 'mean' flow. However, if it is desired to invoke the symmetry of the stress, in the context of the present theory it can be accommodated by simply setting

$$\partial\psi/\partial d_{i,j} = 0, \quad \nu_1 = \nu_2 = \nu_4 = 0. \tag{6.29}$$

7. Turbulent–non-turbulent interfaces

The jump conditions across an interface separating a turbulent fluid and a non-turbulent material (such as a laminar fluid or an elastic medium) can be obtained as a special case of the discontinuity conditions derived in §3. Drastic changes in the flow rates and flow structure, on a microscopic scale, are observed to occur very close to these interfaces (Clauser 1956, p. 6). For the macroscopic theory of the present paper, regions of rapid change adjacent to an interface are considered as part of the interface and manifest themselves through the surface supply terms introduced in §3. Setting the mass supply M equal to zero in (3.4), the relevant jump conditions across a surface of discontinuity in a turbulent flow which follow from (3.4)–(3.9) consist of

$$[\![\rho \boldsymbol{u}\cdot\boldsymbol{v}]\!] = 0, \quad [\![(\rho\eta_{\mathrm{H}}\boldsymbol{u}+\boldsymbol{p}_{\mathrm{H}})\cdot\boldsymbol{v}]\!] = E_{\mathrm{H}}, \quad [\![(\rho\eta_{\mathrm{T}}\boldsymbol{u}+\boldsymbol{p}_{\mathrm{T}})\cdot\boldsymbol{v}]\!] = E_{\mathrm{T}}, \quad [\![\rho_{\mathrm{E}}\boldsymbol{u}\cdot\boldsymbol{v}]\!] = M_{\mathrm{E}}, \tag{7.1}$$

as well as the four jump conditions (3.5)–(3.8) without change, where the surface supply terms on the right-hand sides of $(7.1)_{2,3,4}$ correspond to E^N in (3.9) for $N = 1, 2, 3$ (i.e.)

$$E_{\mathrm{H}} = E^1, \quad E_{\mathrm{T}} = E^2, \quad M_{\mathrm{E}} = E^3. \tag{7.2}$$

In certain instances, many of the other surface supply terms may also be set to zero. However, because it is desirable to allow the director $\boldsymbol{d}$ to be discontinuous across an interface separating a turbulent fluid and a non-turbulent material, it may be too restrictive to set $\boldsymbol{F}$, $\boldsymbol{L}$, and $\boldsymbol{M}$ all equal to zero.

We now proceed to obtain a set of jump conditions appropriate for turbulent–non-turbulent interfaces by incorporating into the general jump conditions the requirement that the effect of θ_{T}, ρ_{E}, $\boldsymbol{d}$ in all constitutive response functions must vanish on the non-turbulent side of the interface. Thus, in line with the notation introduced following (3.2), let the value of any function ϕ be denoted as ϕ_2 on the turbulent side and as ϕ_1 on the non-turbulent side of the interface. Then the eight jump conditions $(7.1)_1$, (3.5)–(3.8) and $(7.1)_{2,3,4}$ reduce to:

$$\left.\begin{aligned}
&(\rho\boldsymbol{u}\cdot\boldsymbol{v})_2 = (\rho\boldsymbol{u}\cdot\boldsymbol{v})_1,\\
&[(\rho\boldsymbol{u})_2\cdot\boldsymbol{v}]\,[\boldsymbol{v}_2-\boldsymbol{v}_1+(y_1\boldsymbol{w})_2]-(\boldsymbol{t}_2-\boldsymbol{t}_1) = \boldsymbol{F},\\
&[(\rho\boldsymbol{u})_2\cdot\boldsymbol{v}]\,(y_1\boldsymbol{v}+y_2\boldsymbol{w})_2-\boldsymbol{m}_2 = \boldsymbol{L},\\
&(\boldsymbol{d}\times\boldsymbol{L})_2 = \boldsymbol{M}-\boldsymbol{x}\times\boldsymbol{F},\\
&(\rho\boldsymbol{u})_2\cdot\boldsymbol{v}[[\epsilon+\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}]]+(y_1\boldsymbol{v}\cdot\boldsymbol{w}+\tfrac{1}{2}y_2\boldsymbol{w}\cdot\boldsymbol{w})_2]-[\boldsymbol{t}\cdot\boldsymbol{v}]-(\boldsymbol{m}\cdot\boldsymbol{w})_2+[\boldsymbol{q}_{\mathrm{H}}\cdot\boldsymbol{v}]+(\boldsymbol{q}_{\mathrm{T}}\cdot\boldsymbol{v})_2 = \Phi,\\
&(\rho\boldsymbol{u})_2\cdot\boldsymbol{v}[\eta_{\mathrm{H}}]+[\boldsymbol{p}_{\mathrm{H}}\cdot\boldsymbol{v}] = E_{\mathrm{H}},\\
&(\rho\boldsymbol{u}\cdot\boldsymbol{v}\eta_{\mathrm{T}})_2+(\boldsymbol{p}_{\mathrm{T}}\cdot\boldsymbol{v})_2 = E_{\mathrm{T}},\\
&(\rho_{\mathrm{E}}\boldsymbol{u})_2\cdot\boldsymbol{v} = M_{\mathrm{E}}.
\end{aligned}\right\}\quad(7.3)$$

8. The inviscid theory of turbulence

We specialize in this section the constitutive theory of §§5 and 6 to an idealized case of a (rate-independent) inviscid turbulent fluid. The designation 'inviscid' here is intended to imply only that the dependence of all response functions on the rate-type kinematical variables A_{ij}, h_i, and h_{ij} is suppressed; the dependence on the rate of turbulent temperature $\dot{\theta}_{\mathrm{T}}$, however, is retained. The fluid is, of course, considered to be inviscid only on the macroscopic scale, so that viscous dissipation of the turbulent fluctuations is still assumed to occur on the microscopic level.

The appropriate constitutive equations for an inviscid turbulent fluid, which follow from various results in (6.9)–(6.11) and (5.33), may be written as

$$\left.\begin{aligned}
&\hat{k}_i = \rho\frac{\partial\psi}{\partial d_i},\quad \hat{m}_{ij} = \rho\frac{\partial\psi}{\partial d_{i,j}},\\
&\hat{t}_{(ij)} = -\left(\rho\rho_{\mathrm{E}}\frac{\partial\psi}{\partial\rho_{\mathrm{E}}}+\rho^2\frac{\partial\psi}{\partial\rho}\right)\delta_{ij}-\tfrac{1}{2}\rho\left(\frac{\partial\psi}{\partial d_{k,j}}d_{k,i}+\frac{\partial\psi}{\partial d_{k,i}}d_{k,j}\right)\\
&\hat{t}_{[ij]} = \tfrac{1}{2}\rho\left(\frac{\partial\psi}{\partial d_j}d_i-\frac{\partial\psi}{\partial d_i}d_j+\frac{\partial\psi}{\partial d_{j,k}}d_{i,k}-\frac{\partial\psi}{\partial d_{i,k}}d_{j,k}\right),
\end{aligned}\right\}\quad(8.1)$$

and

$$\left.\begin{aligned}
&p_{\mathrm{H}i} = P^0 g_{\mathrm{H}i},\quad p_{\mathrm{T}i} = F^0 g_{\mathrm{T}i},\quad \xi_{\mathrm{E}} = H^0\dot{\theta}_{\mathrm{T}},\\
&\xi_{\mathrm{H}} = E^0-\frac{1}{\rho\theta_{\mathrm{H}}}P^0 g_{\mathrm{H}i}g_{\mathrm{H}i},\\
&\xi_{\mathrm{T}} = -\frac{\theta_{\mathrm{H}}}{\theta_{\mathrm{T}}}E^0-\frac{1}{\rho\theta_{\mathrm{T}}}F^0 g_{\mathrm{T}i}g_{\mathrm{T}i}.
\end{aligned}\right\}\quad(8.2)$$

The specific Helmholtz free energy ψ in (8.1) and the coefficients P^0, F^0, H^0 and E^0 in (8.2) depend on the set of variables (6.3). These response coefficients are restricted only by the invariance conditions under superposed rigid body motions; however, the free energy ψ must satisfy equation $(6.10)_3$ as well.

We consider now the special case discussed in §6 in which ψ and E^0 are assumed to be linear

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in $d_{i,j}$ and quadratic in d_i, while F^0, H^0 and P^0 are assumed to be only quadratic in d_i. The invariance conditions and the restriction $(6.10)_3$ then demand that

$$\left.\begin{aligned} &\psi = \psi_0+\psi_1 d_{i,i}+\psi_2 d_i d_i, \quad H^0 = \gamma_0+\gamma_1 d_i d_i, \\ &P^0 = -\kappa_0-\kappa_1 d_i d_i, \quad F^0 = -\lambda_0-\lambda_1 d_i d_i, \\ &E^0 = \phi_0+\phi_1 d_{i,i}+\phi_2 d_i d_i, \end{aligned}\right\} \tag{8.3}$$

where the coefficients on the right-hand sides of $(8.3)_{1-5}$ are functions of the variables (6.13). Substituting (8.3) into (8.1) and (8.2), the constitutive equations for a compressible inviscid turbulent fluid reduce to

$$\left.\begin{aligned} &\hat{k}_i = 2\rho\psi_2 d_i, \quad \hat{m}_{ij} = \rho\psi_1\delta_{ij}, \\ &\hat{t}_{ij} = -\left(\rho\rho_{\mathrm{E}}\frac{\partial\psi}{\partial\rho_{\mathrm{E}}}+\rho^2\frac{\partial\psi}{\partial\rho}\right)\delta_{ij}-\rho\psi_1 d_{j,i}, \\ &p_{\mathrm{H}i} = -(\kappa_0+\kappa_1 d_j d_j)\, g_{\mathrm{H}i}, \quad p_{\mathrm{T}i} = -(\lambda_0+\lambda_1 d_j d_j)\, g_{\mathrm{T}i}, \\ &\xi_{\mathrm{E}} = (\gamma_0+\gamma_1 d_i d_i)\,\dot{\theta}_{\mathrm{T}}, \\ &\xi_{\mathrm{H}} = (\phi_0+\phi_1 d_{i,i}+\phi_2 d_i d_i)+(\kappa_0+\kappa_1 d_j d_j)\frac{g_{\mathrm{H}i}g_{\mathrm{H}i}}{\rho\theta_{\mathrm{H}}}, \\ &\xi_{\mathrm{T}} = -\frac{\theta_{\mathrm{H}}}{\theta_{\mathrm{T}}}(\phi_0+\phi_1 d_{i,i}+\phi_2 d_i d_i)+(\lambda_0+\lambda_1 d_j d_j)\frac{g_{\mathrm{T}i}g_{\mathrm{T}i}}{\rho\theta_{\mathrm{T}}}, \end{aligned}\right\} \tag{8.4}$$

and we note that for an incompressible fluid $(8.4)_3$ must be replaced by

$$t^*_{ij}+\tilde{t}_{ij} = -p\delta_{ij}-\rho\psi_1 d_{j,i}, \tag{8.5}$$

where $p = p(\boldsymbol{x}, t)$ is a Lagrange multiplier.

With the use of the constitutive relations $(8.3)_1$ and (8.4), the governing equations for a compressible inviscid turbulent fluid can be displayed as

$$\dot{\rho}+\rho\frac{\partial v_i}{\partial x_i} = 0, \tag{8.6}$$

$$\begin{aligned} \rho(\dot{v}_i+y_1\dot{w}_i) = \rho b_i-\frac{\partial}{\partial x_i}\left(\rho\rho_{\mathrm{E}}\frac{\partial\psi}{\partial\rho_{\mathrm{E}}}+\rho^2\frac{\partial\psi}{\partial\rho}\right) \\ -\frac{\partial}{\partial x_j}\left(\rho\psi_1\frac{\partial d_j}{\partial x_i}\right)+\frac{\partial}{\partial x_j}\left\{d_i\left[\rho l_j+\frac{\partial}{\partial x_j}(\rho\psi_1)-2\rho\psi_2 d_j-\rho(y_1\dot{v}_j+y_2\dot{w}_j)\right]\right\}, \end{aligned} \tag{8.7}$$

$$\rho(y_1\dot{v}_i+y_2\dot{w}_i)\, d_i = \rho l_i d_i+\frac{\partial}{\partial x_i}(\rho\psi_1)\, d_i-2\rho\psi_2 d_i d_i, \tag{8.8}$$

$$\dot{\rho}_{\mathrm{E}}+\rho_{\mathrm{E}}\frac{\partial v_i}{\partial x_i} = (\gamma_0+\gamma_1 d_i d_i)\,\dot{\theta}_{\mathrm{T}}, \tag{8.9}$$

$$\rho\dot{\eta}_{\mathrm{H}} = \rho s_{\mathrm{H}}+\frac{\partial}{\partial x_i}\left[(\kappa_0+\kappa_1 d_l d_l)\frac{\partial\theta_{\mathrm{H}}}{\partial x_i}\right]+\rho\left(\phi_0+\phi_1\frac{\partial d_i}{\partial x_i}+\phi_2 d_i d_i\right)+\frac{1}{\theta_{\mathrm{H}}}(\kappa_0+\kappa_1 d_l d_l)\frac{\partial\theta_{\mathrm{H}}}{\partial x_i}\frac{\partial\theta_{\mathrm{H}}}{\partial x_i}, \tag{8.10}$$

$$\rho\dot{\eta}_{\mathrm{T}} = \rho s_{\mathrm{T}}+\frac{\partial}{\partial x_i}\left[(\lambda_0+\lambda_1 d_l d_l)\frac{\partial\theta_{\mathrm{T}}}{\partial x_i}\right]-\frac{\rho\theta_{\mathrm{H}}}{\theta_{\mathrm{T}}}\left(\phi_0+\phi_1\frac{\partial d_i}{\partial x_i}+\phi_2 d_i d_i\right)+\frac{1}{\theta_{\mathrm{T}}}(\lambda_0+\lambda_1 d_l d_l)\frac{\partial\theta_{\mathrm{T}}}{\partial x_i}\frac{\partial\theta_{\mathrm{T}}}{\partial x_i}. \tag{8.11}$$

Also, the entropies η_H and η_T are given by $(6.10)_1$ and (6.15), respectively. Equations (8.6)–(8.11) represent, respectively, the consequences of the conservation of mass, the conservation of (classical) linear momentum, the conservation of director momentum, the balance of eddy density, the balance of thermal entropy and the balance of turbulent entropy. The coefficients ψ_0, ψ_1, and ψ_2 and the various coefficients $\gamma_0, \gamma_1, \ldots \phi_0, \phi_1, \phi_2$ on the right-hand sides of (8.9)–(8.11) are functions of the variables (6.13). (We do not consider here restrictions on these coefficients which may result from Second Law type statements.)

The governing equations for incompressible inviscid turbulence are identical to the system (8.6)–(8.11) with the following changes:

(1) the mass conservation equation (8.6) is satisfied identically by the incompressibility constraint

$$v_{i,i} = 0, \tag{8.12}$$

leading to ρ = constant;

(2) the additional term $-\partial p/\partial x_i$ due to constrained response, where $p = p(x, t)$ is a Lagrange multiplier, is added to the right-hand side of (8.7);

(3) all coefficients in (8.6)–(8.11) are in this case independent of ρ. It is evident that even for inviscid flow, the governing equations for a turbulent fluid are fairly complex. In order to more readily discuss certain features of these equations, we now introduce a number of plausible simplifying assumptions for the internal energy ϵ and the thermal and turbulent entropies η_H and η_T, namely

$$\epsilon = c_H \theta_H + c_T \theta_T, \quad \eta_H = \eta_H(\theta_H), \quad \eta_T = \eta_T(\rho_E, \theta_T, d_i, d_{i,j}). \tag{8.13}$$

Here c_H and c_T are constant thermal and turbulent specific heats, respectively; and we emphasize that while η_H is taken to depend only on θ_H, η_T is assumed to be independent of θ_H but is a function of the remaining variables $\mathscr{V}_0$ in (6.3) except ρ, which is now a constant. For an incompressible fluid with the use of (4.10), $(6.10)_1$, (6.15), $(8.3)_1$ and the assumptions (8.13), the thermal and turbulent entropy equations (8.10) and (8.11) are:

$$\frac{\rho c_H}{\theta_H}\dot{\theta}_H = \rho s_H + \frac{1}{\theta_H}\frac{\partial}{\partial x_i}\left[\theta_H(\kappa_0 + \kappa_1 d_i d_i)\frac{\partial \theta_H}{\partial x_i}\right] + \rho(\phi_0 + \phi_1 d_{i,i} + \phi_2 d_i d_i), \tag{8.14}$$

$$\begin{aligned}\frac{\rho c_T}{\theta_T}\dot{\theta}_T = \rho s_T + \frac{1}{\theta_T}\frac{\partial}{\partial x_i}\left[\theta_T(\lambda_0 + \lambda_1 d_i d_i)\frac{\partial \theta_T}{\partial x_i}\right] \\ -\rho\frac{\theta_H}{\theta_T}(\phi_0 + \phi_1 d_{i,i} + \phi_2 d_i d_i) + \frac{2\rho\psi_2}{\theta_T} d_i w_i + \frac{\rho\psi_1}{\theta_T}(w_{i,i} - v_{k,i} d_{i,k}).\end{aligned} \tag{8.15}$$

From the form of (8.14) and (8.15), it is evident that thermal and turbulent heat conductivities k_H and k_T can be defined as

$$k_H = \theta_H(\kappa_0 + \kappa_1 d_i d_i), \quad k_T = \theta_T(\lambda_0 + \lambda_1 d_i d_i). \tag{8.16}$$

Also, from (4.10), $(6.10)_1$, (6.15) and the assumptions (8.13), it follows that γ_1 must vanish and the coefficient functions ψ_0, ψ_1 and ψ_2 must satisfy the equations

$$\left.\begin{aligned}\psi_0 &= \epsilon + \theta_H\frac{\partial \psi_0}{\partial \theta_H} + \theta_T\frac{\partial \psi_0}{\partial \theta_T} + \gamma_0\theta_T\frac{\partial \psi_0}{\partial \rho_E}, \\ \psi_1 &= \theta_T\frac{\partial \psi_1}{\partial \theta_T} + \gamma_0\theta_T\frac{\partial \psi_1}{\partial \rho_E}, \\ \psi_2 &= \theta_T\frac{\partial \psi_2}{\partial \theta_T} + \gamma_0\theta_T\frac{\partial \psi_2}{\partial \rho_E}.\end{aligned}\right\} \tag{8.17}$$

Given γ_0, the expressions (8.17) help to restrict the possible forms of the constitutive equations for ψ_0, ψ_1 and ψ_2.

It should be noted that the mechanical power P does not necessarily vanish for an inviscid incompressible turbulent fluid. Also, the off-diagonal components of t_{ij} do not necessarily vanish for an inviscid turbulent fluid, nor are the diagonal components necessarily all equal. The terms involving the coefficients ϕ_0, ϕ_1 and ϕ_2 in (8.14) and (8.15) originally occur in the expressions for ξ_H and ξ_T and are associated with the dissipation of turbulent fluctuations into thermal heat. We might, therefore, expect these coefficients to be non-negative. Other terms in (8.14) involving κ_0 and κ_1 are associated with the diffusion of thermal energy, while similar terms in (8.15) involving λ_0 and λ_1 are related to the diffusion of turbulent fluctuations in the medium. Because it is expected that both thermal and turbulent heat diffuse in the direction of decreasing (either thermal or turbulent) temperature, the coefficients κ_0, κ_1, λ_0 and λ_1 are also expected to be non-negative. Further, for an incompressible fluid, it might be expected that the eddy density, ρ_E will increase whenever θ_T increases, thereby implying that the coefficients γ_0 and γ_1 of the rate of eddy production ξ_E are also non-negative.

The last two terms in (8.15) may be associated with the production of turbulent fluctuations due to straining represented by w, its gradient and gradient of $\boldsymbol{v}$. These terms are obtained from the turbulent entropy rate $\dot{\eta}_T$ in (8.11) after using (6.15) and the form $(8.3)_1$ for the free energy ψ. If ψ is assumed to depend on higher orders of d_i and $d_{i,j}$ than that assumed in $(8.3)_1$, or if certain viscous effects (discussed in §6) are included in the analysis, additional turbulent production terms must necessarily be included in (8.15). Although one might expect the director to increase as the rate of shearing of the fluid increases, this effect cannot be included in the present inviscid theory because of the fact that the intrinsic director force $\hat{k}_i$ is assumed to be independent of A_{ij}. However, the jump condition for director momentum does admit a source (or supply) term which can be expressed as a function of the relative velocities on either side of the jump. It may be recalled in this connection, that the so-called 'inner' or 'near-wall' region of the microscopic turbulent boundary layer (where much of the production of turbulent fluctuation and alignment of large eddies occurs) is included in the jump conditions at the surface of discontinuity in the macroscopic theory discussed here.

In summary, we note that the inviscid assumption leads to the following features:

(1) the dissipation of the turbulent fluctuations into thermal heat;

(2) the diffusion of both thermal heat and turbulent fluctuations;

(3) the possibility of shear force, represented by the off-diagonal components of the stress tensor, on a wall (even in the absence of a velocity gradient) because of the presence of the director;

(4) an increase in the eddy density because of increase in the intensity of turbulent fluctuations;

(5) the possibility of non-equal diagonal components of the stress tensor t_{ij} even for a vanishing velocity gradient; and

(6) the production of turbulent fluctuations by the rate of deformation of the fluid, occurring both within the fluid and along a surface of discontinuity.

Clearly the system of equations (8.7)–(8.11) would not be applicable to turbulent flows when the turbulent fluctuations become weak and the effects of rate-dependent terms involving the variables (6.4) become increasingly more important. With this observation, it seems reasonable to assume that when the turbulent fluctuations become sufficiently intense, the inviscid equations (8.4) and (8.5) may be sufficient to describe the behaviour of many macroscopic turbulent flows (or at least certain parts of these flows). Below this level of intensity, however, the more general constitutive relations (6.14)–(6.23), or possibly similar relations which may depend on higher orders of d_i and $d_{i,j}$, must be used. That the latter equations do indeed approach the equations appropriate for laminar flow when the additional turbulent variables $(\theta_T, \boldsymbol{d}, \rho_E)$ vanish can be readily demonstrated. The exact domain of applicability of the system of equations of this section can, of course, only be determined by comparison of predictions of the theory with experimental results.

9. Additional remarks

The large eddies of turbulent flow are known to possess a great number of shapes and orientations (see Grant (1958), Townsend (1956, 1976), Mumford (1982), Savill (1987)). Any model describing the effect of these structures on the macroscopic flow must necessarily simplify the complicated responses and interactions of these large eddies to obtain a manageable set of equations. Also, it is not always clear whether or not a given set of eddies (such as the 'vortex street' type eddies in turbulent wake flow) represents a background structure on the microscopic level or could be interpreted as a (non-chaotic) solution of the governing equations on the macroscopic level. Indeed, the identification of the 'macroscopic' flow is very much dependent on the type of modelling employed. For example, in the 'large eddy simulation' technique described by Reynolds (1976), only the smallest scales of turbulent fluctuations are considered 'microscopic' (i.e. requiring turbulence modelling) and all large-scale fluctuations are considered 'macroscopic' (i.e. are directly computed). The present theory, therefore, does *not* neglect the large vortex street type eddies, or similar other very large eddy structures, but merely includes them as part of the macroscopic flow. In the present paper, we have attempted to construct a physically motivated and a relatively simple theory which seems to incorporate most of the observed features of turbulent fluid flow. The choice for the mechanical aspects of the model introduced in §5 is influenced by a thorough review of the experimental literature and examination of the macroscopic responses arising from consideration of several possible (director) constraints; and these, in turn, seem to indicate that the alignment of the large eddy vorticity on the microscopic level with the macroscopic rate of deformation tensor (in the manner described in §5) is largely responsible for fluid anisotropy. It may also be noted that identification of the exact form of the microscopic structures causing macroscopic flow anisotropy (whether these structures have the form of 'roller' eddies, 'hairpin' eddies, etc.) is not as important for the theory developed here as is identification of the *mechanism* causing the anisotropy (i.e. straining of the eddies by the macroscopic flow), which is the basis of the constraint $(5.15)_1$. Also, we note that the effect of alignment of eddies at recent past times before the current time t on the macroscopic flow may become important during rapid large changes in the rate of deformation tensor $(5.1)_4$. The latter effect has been implicitly neglected in the present paper, in line with the statement in Appendix A (see statement (b) of Townsend 1956).

It should be emphasized again that the constitutive equations utilized in obtaining the standard Navier–Stokes equations are not sufficient to adequately describe turbulent flow on a macroscopic level. The validity of this remark is most evident in the inability of the Navier–Stokes equations to properly account for fluid 'anisotropy'. In particular, numerous experiments (for example, Uberoi 1956, Tucker & Reynolds 1968) have determined that for a uniform flow in the positive x-direction, the t_{11} component of the Cauchy stress tensor may be significantly larger than the t_{22} and t_{33} components. Such an effect can be clearly accommodated for by the present theory (see, for example, equations $(5.23)_1$ and (8.5)). The Navier–Stokes equations, which demand that all three components of the stress tensor be equal for such a flow, obviously cannot encompass such effects. Even in attempting to describe 'isotropic' turbulence, most theoretical analyses have had to cope with the introduction of additional quantities pertaining to the turbulent motion (such as 'mixing length,' 'turbulent kinetic energy'). The assumption that the various response functions depend upon these additional quantities implies that new constitutive assumptions have been introduced, even though such constitutive relations are generally assumed to be similar to those used for laminar flows.

Finally, we need to make some remarks in regard to the applicability and appropriate use of the invariance requirements under superposed rigid body motions (SRBM) pertaining to theories of turbulent flow. It is conceivable that disregard of invariance may be inconsequential in a particular application or context; however, it plays an important role in the development of general theories of the type under discussion, including of course some restrictions on the constitutive equations. We must, therefore, contest certain remarks which have appeared in the literature (Lumley 1970, 1983) implying that invariance requirements do not apply to turbulent flow. (Actually, Lumley refers to the 'principle of material frame indifference,' rather than requirements of invariance under SRBM. The difference between the two is briefly discussed in the last paragraph of §2.) Lumley's argument is based on the claim that the equations of motion in neither the continuum nor molecular theories are invariant. This claim is false on both counts; in this connection, see the discussions by Naghdi (1972, pp. 484–486) in the context of (three-dimensional) continuum mechanics, by Green (1982) in the contexts of both continuum mechanics and molecular theory, and Woods's (1983, p. 432) acknowledgement of the importance of invariance in the latter category. Further, Lumley's statement (Lumley 1983, p. 1100, para. 2, lines 1–4) to the effect that the requirements of invariance under SRBM on constitutive equations is equivalent to '...ignoring the inertia of any motion responsible for the development of stress' is also incorrect. This is because any change in inertia due to a rigid motion is exactly balanced by a change in the body force such that the momentum equation as a whole remains invariant.

As noted previously (§§2 and 6), the various thermomechanical fields that occur in the balance laws of the present paper are required to satisfy appropriate invariance conditions under superposed rigid body motions. These requirements place certain restrictions on the constitutive equations; and, as noted by Speziale (1979, 1980), are not satisfied in many developments found in the literature.

The results reported here were obtained in the course of research supported by the U.S. Office of Naval Research under contract N00014-86-K-0057, Work Unit 4322 534 with the University of California, Berkeley.

Appendix A

The purpose of this appendix is to collect a number of quotations from the literature (arranged in alphabetical order by names of the authors) on turbulent flow in order to provide support for the *model* proposed in §5 of the paper. For each listing after indicating the source, before the actual quotation, we list the page number, paragraph, and the relevant lines of the quotation. Multiple quotations from the same source are grouped under the same reference heading. (Note: The term 'strain', which appears in many of the quotations, evidently is intended for strain rate or more generally the rate of deformation tensor.)

Grant, H. L. (1958)

(*a*) p. 171, para. 2, lines 1–4

In a plane wake, the nature of the strain is a combination of a rotation and a plain strain with principal axes of strain in directions at 45° to the usual coordinates, i.e., to the mean direction of flow. The resultant anisotropy of turbulent intensity is also greatest for axes aligned approximately in these directions.

(*b*) p. 183, para. 4, lines 1–3

...the large scale motions in the boundary layer are essentially the same as those which have been found in the wake...

(*c*) p. 189, para. 3, lines 3–4

...it is hard to account for the very slow decrease of anisotropy after the distortion.

Krishnamurti, R. & Howard, L. N. (1981)

(*a*) p. 1981, para. 1, lines 1–10

In a horizontal layer of fluid heated from below and cooled from above, cellular convection with horizontal length scale comparable to the layer depth occurs for small enough values of the Rayleigh number. As the Rayleigh number is increased, cellular flow disappears and is replaced by a random array of transient plumes. Upon further increase, these plumes drift in one direction near the bottom and in the opposite direction near the top of the layer with the axis of plumes tilted in such a way that horizontal momentum is transported upward via the Reynolds stress.

(*b*) p. 1985, para. 3, lines 3–8

The small-scale plumes were considerably narrower than the depth d of the layer, but they tilted approximately 45° [to the direction of flow] and in this sense they occupied a horizontal distance of about d. The large-scale flow was observed to continue around the annulus, in one direction near the bottom and in the opposite direction near the top.

Mumford, J. C. (1982)

p. 241, para. 3, lines 1–4

The results indicate that the large eddies in the fully turbulent regime of the flow are roller-like structures with axes aligned approximately either with the direction of the strain associated with the mean velocity gradient or with the direction of homogeneity (spanwise).

Rivlin, R. S. (1957)

p. 214, para. 5, lines 6–8

The eddies in a turbulent Newtonian fluid will presumably undergo preferential orientation when the turbulent fluid is sheared providing a possible mechanism for the effects in the turbulent fluid.

Rogers, M. M. & Moin, P. (1987)

p. 33, para. 1, lines 1–4, 14–16, 18–20

The structure of the vorticity fields in homogeneous turbulent shear flow and various irrotational straining flows is examined using results from direct numerical simulations of the unsteady, incompressible Navier–Stokes equations with up to 128 × 128 × 128 grid points ... Examination of irrotational axisymmetric contraction, axisymmetric expansion, and plane strain flows shows, as expected, that the vorticity tends to be aligned with the direction of positive strain ... The simulations strongly indicate that the vorticity occurs in coherent filaments that are stretched and stengthened by the mean strain.

Savill, A. M. (1987)

p. 535–536, para. 5, lines 1–7

Consideration of the equation for turbulent vorticity in the presence of mean shear then indicates that this has two effects on a vortex element: A mean vortex line can be stretched along its length, generating vorticity fluctuations, while the vertical component of the turbulent vorticity is rotated and stretched by the mean motion, generating a streamwise component. A combination of lifting, shearing, and stretching thus produces a horseshoe vortex loop ...

Taylor, G. I. & Green, A. E. (1937)

p. 502, para. 3, lines 1–8

At the outset the extreme limitations of mathematical methods are very evident, for it is only in special cases where the initial motion is such that one of the essential features of turbulent motion (i.e., extension along vortex lines) is absent that the subsequent motion has so far been calculated. By far the largest class of fields of flow which has been analyzed mathematically is two-dimensional. Since the vortex lines are then perpendicular to the plane of motion, they are not extending, and this essential characteristic of turbulent flow is therefore absent.

Townsend, A. A. (1956)

(*a*) p. 101, para. 6, lines 1–7

The large eddies, which distort the bounding surface, are simple eddies with central vorticity along the principal axis of positive mean rate of strain, elongated in the direction of flow and centered near the plane of maximum rate of shear. Their life is comparable with the time for appreciable development of the flow, but they are not permanent structures, new ones arising as old ones disappear.

(*b*) p. 117, para. 2, lines 11–15

It is expected that a turbulent shear flow with an equilibrium structure will respond to a superimposed velocity perturbation by establishing quite quickly the equilibrium structure appropriate to the new type of strain and oriented along the new principal axes.

(*c*) p. 120, para. 2, lines 1–5

The large eddies are believed to begin their distinct existence as a chance configuration of the turbulent motion of suitably large scale and orientation that it can absorb energy from the mean flow in sufficient quantity to prevent its rapid disappearance by turbulent transfer.

(*d*) p. 48, para. 2, lines 1–2

When anisotropic turbulence is produced ... the return to isotropy is found to be very slow ...

Townsend, A. A. (1976)

(*a*) p. 187, para. 2, lines 1–12

Near identity of the directions of the eddies could be a result either of the process of generation or of a mechanism for alignment of the developed eddies. To the extent that the active portion of each eddy is to some degree concentrated near its centre, the generation process should lead to a normal double-roller eddy with shear stress and axis aligned with the velocity gradient. Further, the inclined and trailing form of the double roller means that fully developed eddies will be kept in alignment by the velocity gradient. Naturally, if directions and magnitudes of shear change rapidly with height, eddies may be distorted or not aligned with the local shear, but the effect is essentially similar to that of energy diffusion in an attached eddy and is expected to be quite small.

(*b*) p. 235, para. 1, lines 6–8

One effect of the mean velocity gradient is to align eddies with their axes parallel to the direction of extension...

(*c*) p. 240, para. 2, lines 4–6

...the behaviour of the displacement and intermittency correlations is what might be expected if roller eddies with axes aligned with the flow performed the distortion.

Tucker, H. J. & Reynolds, A. J. (1968)

(*a*) p. 669, para. 9, lines 1–2

The results indicate that the structure developed in the turbulence is mainly the result of the mean motion straining the eddies...

(*b*) p. 669, para. 10, lines 1–3

...the influence of uniform strain on the structure of turbulence depends primarily on the strain and to a small extent, if at all, on the orientation of the strain relative to the main flow direction.

Uberoi, M. S. (1956)

p. 764, para. 2, lines 13–18

Naturally, directional influence of shear flow on the turbulence is most pronounced in the direction of the principal axes. In shear flow, measurement of mean square turbulent velocity gradients along the principal axis of the rate of deformation might reveal anisotropy.

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